ON SOME NEW DISCRETE INEQUALITIES INVOLVING PARTIAL SUMS

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The purpose of this paper is to obtain some new discrete inequalities of the Gronwall type involving partial sums by using the discrete inequality of Cheng-Ming Lee. The inequalities herein yield as special cases several additional new discrete inequalities.

1. INTRODUCTION

Discrete analogues of Opial's inequality (1960) and its generalizations were given by Beesack (1969) Lee (1968) and Wong (1967). Lee (1968) has given a discrete analogue of the inequality of Yang (1966). A suitable version of Lee's inequality may be stated as follows.

Lemma 1 (Lee 1968) — Let $y(n)$ for $n = 0, 1, 2, \ldots$ be a non-decreasing sequence of nonnegative numbers, and let $y(0) = 0$. Then we have

$$\sum_{s=0}^{n-1} [\triangle y(s)]^q \, y^p(s+1) \leqslant \frac{qn^p}{p+q} \sum_{s=0}^{n-1} [\triangle y(s)]^{p+q}$$

for $p \geqslant 1$, $q \geqslant 1$, and $n = 0, 1, 2, \ldots$, where $\triangle y(s) = y(s+1) - y(s)$.

This inequality also appeared in Mitrinovic (1970) which includes the discrete inequality given by Wong (1967). Our aim in the present paper is to use the discrete inequality established in Lemma 1 to obtain some new discrete inequalities involving partial sums which can be used in some problems in the theory of finite difference equations. For several interesting and useful discrete inequalities and their applications we refer the interested reader to the author's papers (Pachpatte 1973a, b; 1977a, b) and some of the references given therein.

2. MAIN RESULTS

In this section we establish our main results on the discrete inequalities by using Lemma 1. Before giving the main results in this section, we first introduce a few of the basic notions involved in our subsequent discussion. Let N be a set of points $n_0 + k$ ($k = 0, 1, 2, \ldots$), where $n_0 \geqslant 0$ is a given integer. The expression $\sum_{s=n_0}^{n-1} b(s)$

represents a solution of the linear difference equation $\Delta x(n) = b(n)$ for all $n \in N$ under the initial condition $x(n_0) = 0$. It is supposed that $\sum_{s=n_0}^{n_0-1} b(s) = 0$. The expression $\prod_{s=n_0}^{n-1} c(s)$ represents a solution of the linear difference equation $x(n+1) = c(n)\, x(n)$ for all $n \in N$ under the initial condition $x(x_0) = 1$. It is supposed that $\prod_{s=n_0}^{n_0-1} c(s) = 1$.

A useful discrete inequality is embodied in the following theorem.

Theorem 1 — Let $x(n)$ for $n \in N$ be a nondecreasing sequence of nonnegative numbers with $x(0) = 0$, and let $f(n)$, $b(n)$ and $c(n)$ for $n \in N$ be nonnegative sequences of numbers for which the inequality

$$\Delta x(n) \leqslant f(n)\Big[a + x(n) + \sum_{s=0}^{n-1} b(s)\, \Delta x(s) + \sum_{s=0}^{n-1} c(s)\Big(\sum_{t=0}^{s-1} [\Delta x(t)]^q\, x^p(t+1)\Big)\Big] \quad \ldots(1)$$

holds for $n \in N$, where a is a positive constant and $p \geqslant 1$, $q \geqslant 1$. If

$$1 + f(n) + f(n)\, b(n) - \frac{qn^p}{p+q}\, c(n) \geqslant 0$$

and

$$1 - a^{(p+q-1)}(p+q-1) \sum_{s=0}^{n-1} f^{p+q}(s) \prod_{t=0}^{s} \Big[1 + f(t) + f(t)\, b(t) + \frac{qt^n}{p+q}\, c(t)\Big]^{(p+q-1)} > 0 \quad \ldots(2)$$

for all $n \in N$, then

$$\Delta x(n) \leqslant f(n)\Big[a \prod_{s=0}^{n-1} \Big[1 + f(s) + f(s)\, b(s) - \frac{qs^p}{p+q}\, c(s)\Big] + \frac{q}{p+q} \sum_{s=0}^{n-1} s^p c(s)\, Q(s) \prod_{t=s+1}^{n-1} \Big[1 + f(t) + f(t)\, b(t) - \frac{qt^p}{p+q}\, c(t)\Big]\Big] \quad \ldots(3)$$

for all $n \in N$, where

$$Q(n) = \frac{a \prod_{s=0}^{n-1} \left[1 + f(s) + f(s)\,b(s) + \frac{qs^p}{p+q}\,c(s)\right]}{\left[1 - a^{(p+q-1)}(p+q-1) \sum_{s=0}^{n-1} f^{p+q}(s) \prod_{t=0}^{s} \left[1 + f(t) + f(t)\,b(t) + \frac{qt^p}{p+q}\,c(t)\right]^{(p+q-1)}\right]^{1/(p+q-1)}} \quad \ldots(4)$$

for all $n \in N$.

Proof : Define

$$m(n) = a + x(n) + \sum_{s=0}^{n-1} b(s) \triangle x(s) + \sum_{s=0}^{n-1} c(s) \left(\sum_{t=0}^{n-1} [\triangle x(t)]^q x^p(t+1) \right), m(0) = a \quad \ldots(5)$$

then we have

$$\triangle m(n) = \triangle x(n) + b(n) \triangle x(n) + c(n) \left(\sum_{t=0}^{n-1} [\triangle x(t)]^q x^p(t+1) \right). \quad \ldots(6)$$

Using Lemma 1 in (6) we have

$$\triangle m(n) \leqslant \triangle x(n) + b(n) \triangle x(n) + c(n) \left(\frac{qn^p}{p+q} \sum_{t=0}^{n-1} [\triangle x(t)]^{p+q} \right)$$

which in view of (1) implies

$$\triangle m(n) \leqslant f(n)\, m(n) + f(n)\, b(n)\, m(n) + c(n) \left(\frac{qn^p}{p+q} \sum_{t=0}^{n-1} f^{p+q}(t)\, m^{p+q}(t) \right).$$

Adding $\frac{qn^p}{p+q} c(n)\, m(n)$ to both sides of the above inequality we have

$$\triangle m(n) + \frac{qn^p}{p+q} c(n)\, m(n) \leqslant f(n)\, m(n) + f(n)\, b(n)\, m(n) + \frac{qn^p}{p+q} c(n) \left[m(n) + \sum_{t=0}^{n-1} f^{p+q}(t)\, m^{p+q}(t) \right]. \quad \ldots(7)$$

Put

$$u(n) = m(n) + \sum_{t=0}^{n-1} f^{p+q}(t)\, m^{p+q}(t),\ u(0) = m(0) = a. \qquad ...(8)$$

Then

$$\triangle u(n) = \triangle m(n) + f^{p+q}(n)\, m^{p+q}(n). \qquad ...(9)$$

Using $\triangle m(n) \leqslant f(n)\, m(n) + f(n)\, b(n)\, m(n) + \dfrac{qn^p}{p+q} c(n)\, u(n)$ from (7) and $m(n) \leqslant u(n)$ from (8) in (9) we see that the inequality

$$u(n+1) - \left[1 + f(n) + f(n)\, b(n) + \frac{qn^p}{p+q} c(n)\right] u(n) \leqslant f^{p+q}(n)\, u^{p+q}(n) \qquad ...(10)$$

is satisfied for $n \in N$. Define

$$e(n) = \prod_{s=0}^{n-1} \left[1 + f(n) + f(n)\, b(n) + \frac{qn^p}{p+q} c(n)\right]^{-1},\ e(0) = 1,$$

then

$$e(n+1) - e(n) = -\left[f(n) + f(n)\, b(n) + \frac{qn^p}{p+q} c(n)\right] e(n+1). \qquad ...(11)$$

Multiplying by $e(n+1)$ to both sides of (10) and using (11) we obtain

$$u(n+1)\, e(n+1) - u(n)\, e(n) \leqslant f^{p+q}(n)\, [u(n)\, e(n+1)]^{p+q}\, e^{-(p+q-1)}(n+1). \qquad ...(12)$$

Because $u(n)$ is monotone increasing, $e(n)$ is monotone decreasing and $-(p+q) < 0$, we know (*see* Pachpatte 1977a)

$$[u(n)\, e(n+1)]^{-(p+q)} \geqslant z^{-(p+q)}$$

for all values of z between $u(n)\, e(n)$ and $u(n+1)\, e(n+1)$. So if we apply mean value theorem to the function

$$F(z) = \frac{z^{-(p+q-1)}}{-(p+q-1)}$$

we see that

$$\frac{[u(n+1)\, e(n+1)]^{-(p+q-1)} - [u(n)\, e(n)]^{-(p+q-1)}}{-(p+q-1)} \leqslant [u(n)\, e(n+1)]^{-(p+q)} \times [u(n+1)\, e(n+1) - u(n)\, e(n)]. \qquad ...(13)$$

From (12) and (13) we obtain

$$[u(n+1)]\, e(n+1)]^{-(p+q-1)} - [u(n)\, e(n)]^{-(p+q-1)}$$
$$\geqslant -(p+q-1)\, f^{p+q}(n)\, e^{-(p+q-1)}(n+1). \quad \ldots(14)$$

Summing up both sides of (14) from 0 to $n-1$ we obtain

$$[u(n)\, e(n)^{-(p+q-1)} \geqslant a^{-(p+q-1)} - (p+q-1) \sum_{s=0}^{n-1} f^{p+q}(s)\, e^{-(p+q-1)}(s+1). \quad \ldots(15)$$

From (15) we have

$$u(n) \leqslant \frac{a \prod_{s=0}^{n-1} \left[1 + f(s) + f(s)\, b(s) + \frac{qs^p}{p+q} c(s)\right]}{\left[1 - a^{(p+q-1)}(p+q-1) \sum_{s=0}^{n-1} f^{p+q}(s) \prod_{t=0}^{s} \left[1 + f(t) + f(t)\, b(t) + \frac{qt^p}{p+q} c(t)\right]^{(p+q-1)}\right]^{1/(p+q-1)}}$$
$$= Q(n). \quad \ldots(16)$$

Substituting this value of $u(n)$ in (7) we obtain

$$m(n+1) - \left[1 + f(n) + f(n)\, b(n) - \frac{qn^p}{p+q} c(n)\right] m(n) \leqslant \frac{qn^p}{p+q} c(n)\, Q(n).$$

Multiplying by $\prod_{s=0}^{n} \left[1 + f(s) + f(s)\, b(s) - \frac{qs^p}{p+q} c(s)\right]^{-1}$ to both sides of the above inequality and summing up both sides from 0 to $n-1$ we obtain the estimate for $m(n)$ such that

$$m(n) \leqslant a \prod_{s=0}^{n-1} \left[1 + f(s) + f(s)\, b(s) - \frac{qs^p}{p+q} c(s)\right]$$
$$+ \frac{q}{p+q} \sum_{s=0}^{n-1} s^p c(s)\, Q(s) \prod_{t=s+1}^{n-1} \left[1 + f(t) + f(t)\, b(t) - \frac{qt^p}{p+q} c(t)\right].$$

Now substituting this bound for $m(n)$ in (1) we obtain the desired bound in (3).

Another interesting and slightly different version of Theorem 1 may be stated as follows.

Theorem 2 — Let $x(n)$ for $n \in N$ be a nondecreasing sequence of nonnegative numbers with $x(0) = 0$, and let $f(n)$, $b(n)$ and $c(n)$ for $n \in N$ be nonnegative sequences of numbers for which the inequality

$$\Delta x(n) \leqslant f(n)\Big[a + x(n) + \sum_{s=0}^{n-1} b(s)(x(s) + \Delta x(s)) + \sum_{s=0}^{n-1} c(s) \Big(\sum_{t=0}^{s-1} [\Delta x(t)]^{q} x^{p}(t+1)\Big)\Big] \quad \ldots(17)$$

holds for all $n \in N$, where a is a positive constant and $p \geqslant 1$, $q \geqslant 1$. If

$$1 + b(n) + f(n) + b(n) f(n) - \frac{qn^{p}}{p+q} c(n) \geqslant 0$$

and

$$1 - a^{(p+q-1)}(p+q-1)\sum_{s=0}^{n-1} f^{p+q}(s) \prod_{t=0}^{s}\Big[1 + b(t) + f(t) + b(t) f(t) - \frac{qt^{p}}{p+q} c(t)\Big]^{(p+q-1)} > 0 \quad \ldots(18)$$

for all $n \in N$, then

$$\Delta x(n) \leqslant f(n)\Big[a \prod_{s=0}^{n-1}\Big[1 + b(s) + f(s) + b(s) f(s) - \frac{qs^{p}}{p+q} c(s)\Big] + \frac{q}{p+q}\sum_{s=0}^{n-1} s^{p} c(s) R(s) \prod_{t=s+1}^{n-1}\Big[1 + b(t) + f(t) + b(t) f(t) - \frac{qt^{p}}{p+q} c(t)\Big]\Big], \quad \ldots(19)$$

for all $n \in N$, where

$$R(n) = \frac{a \prod_{s=0}^{n-1}\Big[1 + b(s) + f(s) + b(s) f(s) + \frac{qs^{p}}{p+q} c(s)\Big]}{\Big[1 - a^{(p+q-1)}(p+q-1)\sum_{s=0}^{n-1} f^{p+q}(s) \prod_{t=0}^{s}\Big[1 + b(t) + f(t) + b(t) f(t) + \frac{qt^{p}}{p+q} c(t)\Big]^{(p+q-1)}\Big]^{1/(p+q-1)}} \quad \ldots(20)$$

for all $n \in N$.

PROOF : Define

$$m(n) = a + x(n) + \sum_{s=0}^{n-1} b(s)\,(x(s) + \triangle x(s))$$
$$+ \sum_{s=0}^{n-1} c(s) \left(\sum_{t=0}^{s-1} [\triangle x(t)]^q x^p(t+1) \right),\ m(0) = a \qquad ...(21)$$

then we have

$$\triangle m(n) = \triangle x(n) + b(n)\,(x(n) + \triangle x(n))$$
$$+ c(n) \left(\sum_{t=0}^{n-1} [\triangle x(t)]^q x^p(t+1) \right). \qquad ...(22)$$

Using Lemma 1 and then $\triangle x(n) \leqslant f(n)\, m(n)$ from (17) and $x(n) \leqslant m(n)$ from (21) in (22) we have

$$\triangle m(n) \leqslant [b(n) + f(n) + b(n) f(n)]\, m(n)$$
$$+ \frac{qn^p}{p+q} c(n) \sum_{t=0}^{n-1} f^{p+q}(t)\, m^{p+q}(t).$$

The rest of the proof follows by the similar argument as in the proof of Theorem 1. We omit the details.

We note that, Theorems 1 and 2 in the special case when (i) $b(n) = 0$, (ii) $b(n) = 0$ and $q = 1$, (iii) $b(n) = 0$ and $p = 1$, $q = 1$, (iv) $f(n) = 1$, reduce to different discrete inequalities which are new to the literature.

In concluding this paper we note that the discrete inequality established in Theorem 1 can be used to obtain a bound on the solutions of a summary difference equation of the form

$$\triangle y(n) = F\Big[n,\ y(n),\ \sum_{s=0}^{n-1} g(n, s, \triangle y(s)),$$
$$\sum_{s=0}^{n-1} h\left(n, s \sum_{t=0}^{s-1} y^p(t+1)\, k(s, t, \triangle y(t)) \right) \Big] \qquad ...(23)$$

under the following conditions on the functions involved in (23),

$$| F[n, y, u, v] | \leqslant f(n)\,[a + |y| + |u| + |v|] \qquad ...(24)$$

$$| g(n, s, \triangle y(s)) | \leqslant b(s)\, | \triangle y(s) | \qquad ...(25)$$

$$| h(n, s, w) | \leqslant c(s)\, | w | \qquad ...(26)$$

$$| k(s, t, \triangle y(t)) | \leqslant | \triangle y(t) |^q \qquad ...(27)$$

for all $n, s, t \in N$, where $f(n)$, a, $b(s)$ and $c(s)$ are as defined in Theorem 1. By using (24), (25), (26) and (27) in (23) and using Theorem 1 we can obtain the bound on $| \Delta y(n) |$ and consequently the bound on $| y(n) |$. Finally we note that the discrete inequality established in Theorem 2 can be used to obtain a bound on the solution of a slight variant of the summary difference equation given in (23).

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