ON SOME TRANSFORMATIONS OF TRIPLE HYPERGEOMETRIC SERIES $F^{(3)}$—II

by M. A. PATHAN, *Department of Mathematics, Al Fateh University, P.O. Box* 656, *Tripoli, Libya*

(*Received* 19 *January* 1978)

In continuation of a recent paper (Pathan 1978), author obtains some more transformations of generalized triple hypergeometric series $F^{(3)}$ and deduce some formulas for Kampé de Fériet's hypergeometric function of two variables as special cases.

§1. This paper is a continuation of an earlier one by the author (Pathan 1978) in which some transformations of general triple hypergeometric series $F^{(3)}$ introduced by Srivastava (1967, p. 428) have been obtained. We add two more transformation formulas of $F^{(3)}$ and conclude this paper by specializing our main results to obtain certain transformation and reduction formulas of Lauricella and Kampé de Fériet hypergeometric functions. (For definition of $F^{(3)}$, *see* Pathan 1978).

§2. The main transformation formulas to be obtained are

$$F^{(3)}\left[\begin{matrix} - :: a ;-; d : a-d-e; f, a+b-d, c+e+d; b ; \\ a+b :: a+c;-;-: - ; g ;-; \end{matrix} 1, y, \frac{x-1}{x}\right]$$

$$= x^{d} F^{(3)}\left[\begin{matrix} a :: - ;-; d : a-d-e; f, a+b-d, c+e+d;-; \\ a+b :: a+c;-;-: - ; g ;-; \end{matrix} x, y, 1-x\right] \quad \ldots(2.1)$$

$$= (1-y)^{-e} F^{(3)}$$

$$\left[\begin{matrix} - :: a ;-; d : a-d-e; g-f, a+b-d, c+e+d; b; \\ a+b :: a+c;-;-: - ; g ;-; \end{matrix} 1, \frac{y}{y-1}, \frac{x+y-1}{x}\right]. \quad \ldots(2.2)$$

To prove (2.1), we evaluate an integral

$$I = \int_0^\infty e^{-\sigma x} x^{\rho-1} E(\alpha, \beta :: \sigma x)\, E(\lambda, \mu :: Cx)\, {}_1F_1(\gamma, \delta; cx)\, dx$$

$$= \frac{\Gamma(\alpha+\rho+\mu)\,\Gamma(\mu)\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\lambda)\,\Gamma(\alpha+\rho+\lambda)\,\Gamma(\beta+\rho+\mu)}{(\sigma)^\rho\,\Gamma(\alpha+\lambda+\rho+\mu)\,\Gamma(\alpha+\beta+\rho+\mu)}$$

$$F^{(3)}\left[\begin{matrix} - :: \alpha+\rho+\mu\ ; - ; \mu : \alpha\ ; \gamma, \alpha+\rho+\lambda, \beta+\rho+\mu;\ \lambda\ ; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1, \dfrac{c}{\sigma}, \dfrac{C-\sigma}{C} \\ \alpha+\lambda+\rho+\mu :: \alpha+\beta+\rho+\mu ; - ; - : - ;\qquad \delta \qquad ; - ; \end{matrix}\right] \quad \ldots(2.3)$$

where

$\text{Re}\,(C) > \frac{1}{2}\,\text{Re}\,(\sigma)$, $\text{Re}\,(\alpha+\gamma+\lambda) > 0$,

$\text{Re}\,(\beta+\gamma+\lambda) > 0$, $\text{Re}\,(\alpha+\gamma+\mu) > 0$, $\text{Re}\,(\beta+\gamma+\mu) > 0$

and $E(\alpha, \beta :: x)$ is MacRobert's E-function (1962).

(2.3) can be obtained by expanding ${}_1F_1$ in a series of powers of cx, integrating term by term by using a result of MacRobert [1962, p. 397 (117)]

$$\int_0^\infty e^{-t}\, t^{\gamma-1}\, E(\alpha, \beta :: t)\, E(\lambda, \mu :: zt)\, dt$$

$$= \Gamma(\beta)\,\Gamma(\lambda)\,\Gamma(\alpha+\gamma+\lambda)\,\Gamma(\beta+\gamma+\mu)$$

$$\sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)\,\Gamma(\mu+r)\,\Gamma(\alpha+\gamma+\mu+r)}{r!\,\Gamma(\alpha+\beta+\gamma+\mu+r)\,\Gamma(\alpha+\gamma+\mu+\lambda+r)}$$

$${}_2F_1\left(\begin{matrix}\lambda, \mu+r \\ \\ \alpha+\gamma+\lambda+\mu+r\end{matrix} ; \frac{z-1}{z}\right) \quad \ldots(2.4)$$

$\text{Re}\,(z) > \frac{1}{2}$, $\text{Re}\,(\alpha+\gamma+\lambda) > 0$, $\text{Re}\,(\beta+\gamma+\lambda) > 0$, $\text{Re}\,(\alpha+\gamma+\mu) > 0$ and $\text{Re}\,(\beta+\gamma+\mu) > 0$, expanding ${}_2F_1$ in series and interpreting the result in the form of $F^{(3)}$.

Applying a relation (Erdélyi *et al.* 1953, p. 105)

$${}_2F_1\left(\begin{matrix}\alpha, \beta \\ \\ \gamma\end{matrix} ; z\right) = (1-z)^{-\beta}\, {}_2F_1\left(\begin{matrix}\gamma-\alpha, \beta \\ \\ \gamma\end{matrix} ; \frac{z}{z-1}\right) \quad \ldots(2.5)$$

in (2.4), we may write

$$I = \frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\mu)\,\Gamma(\lambda)\,\Gamma(\beta+\rho+\mu)\,\Gamma(\alpha+\rho+\lambda)}{\Gamma(\alpha+\beta+\rho+\mu)\,\Gamma(\alpha+\rho+\mu+\lambda)}(C)^{\mu}(\sigma)^{-\rho-\mu}$$

$$\times F^{(3)}\left[\begin{matrix} \alpha+\rho+\mu :: - ;-;\mu:\alpha;\gamma,\alpha+\rho+\lambda,\beta+\rho+\mu;-; \\ \qquad\qquad\qquad\qquad \frac{C}{\sigma},\frac{c}{\sigma},\frac{\sigma-C}{\sigma} \\ \alpha+\rho+\lambda+\mu::\alpha+\beta+\rho+\mu;-;-:-;\quad \delta \quad ;-; \end{matrix}\right]. \qquad ...(2.6)$$

Equivalence of (2.3) and (2.6) and adjustment of parameters would yield a formula (2.1).

Combining the integral (2.3) with the Kummer's transformation (Slater 1960, p. 12)

$${}_1F_1(\gamma, \delta; cx) = e^{cx}\,{}_1F_1(\delta-\gamma, \delta; -cx) \qquad ...(2.7)$$

using (2.4) and interpreting the result in terms of $F^{(3)}$, we obtain (2.2).

§3. We shall mention some interesting particular cases of (2.2) and (2.3). For $x \to 1$, (2.2) reduces to

$$F^{(2)}\left[\begin{matrix} a : d, a-d-e; f, a-d+b, c+e+d; \\ \qquad\qquad\qquad 1, y \\ a+b, a+c: \quad - \quad ; \quad g \quad ; \end{matrix}\right]$$

$$= (1-y)^{-e}$$

$$\times F^{(3)}\left[\begin{matrix} - :: a ;-; d : a-d-e; g-f, a-d+b, c+d+e; b ; \\ \qquad\qquad\qquad 1, \frac{y}{y-1}, y \\ a+b :: a+c;-;-: \quad - \quad ; \quad g \quad ;-; \end{matrix}\right] \qquad ...(3.1)$$

where $F^{(2)}$ is Kampé de Fériet's hypergeometric function. (For definition of $F^{(2)}$, *see* Pathan 1978).

For $a = d + e$, (2.2) reduces to

$$F^{(2)}\left[\begin{matrix} - : d+e, f, b+e; d, b; \\ \qquad\qquad y, \frac{x-1}{x} \\ b+d+e: \quad g \quad ;-; \end{matrix}\right]$$

$$= (1-y)^{-e}\,F^{(2)}\left[\begin{matrix} - : d+e, g-f, b+e; d, b; \\ \qquad\qquad \frac{y}{y-1}, \frac{x+y-1}{x} \\ b+d+e: \quad g \quad ;-; \end{matrix}\right] \qquad ...(3.2)$$

which further for $x \to 1$, yields

$$F^{(2)}\begin{bmatrix} - : d+e, g-f, b+e; d, b; \\ \qquad\qquad\qquad \dfrac{y}{y-1}, y \\ d+b+e: \quad g \quad ; - ; \end{bmatrix}$$

$$= (1-y)^e \, {}_3F_2\begin{pmatrix} d+e, f, b+e \\ \qquad ; y \\ b+d+e, \; g \end{pmatrix}. \qquad \ldots(3.3)$$

When $d \to 0$, (3.3) reduces to (2.5).

On the other hand, a special case of formula (2.1) when $y \to 0$ leads to

$$F^{(2)}\begin{bmatrix} - : a, a-d-e; d, b; \\ \qquad\qquad 1, \dfrac{x-1}{x} \\ a+b: \quad a+c; - ; \end{bmatrix}$$

$$= x^d \, F^{(2)}\begin{bmatrix} a : a-d-e; d; \\ \qquad\qquad x, 1-x \\ a+b: \quad a+c \; ; - ; \end{bmatrix}. \qquad \ldots(3.4)$$

For $a = d + e$, (3.4) yields (2.5) again.

Our results (2.1) and (2.2) are, furthermore, reducible to Lauricella functions F_S, given by

$$F_S[a_1, a_2, a_2, b_1, b_2, b_3; c_1, c_1, c_1; x, y, z]$$

$$= \sum_{m,n,p} \frac{(a_1)_m (a_2)_{n+p} (b_1)_m (b_2)_n (b_3)_p \, x^m y^n z^p}{(c_1)_{m+n+p} \, m! \, n! \, p!} \qquad \ldots(3.5)$$

in the subsequent notation of Saran (1954). Thus, for $c = 0$ and $g = e + d$, we obtain

$$F_S\left[f, d, d, a+b-d, b, a-d-e; a+b, a+b, a+b; y, \frac{x-1}{x}, 1\right]$$

$$= x^d \, F^{(3)}\begin{bmatrix} a :: -; -; d : a-d-e; f, a+b-d; -; \\ \qquad\qquad\qquad x, y, 1-x \\ a+b :: a; -; -: \quad - \quad ; \quad - \quad ; -; \end{bmatrix}$$

$$\ldots(3.6)$$

$$= (1-y)^{-e}\, Fs\Big[e+d-f, d, d, a-d+b, b, a-d-e; a+b, a+b,$$

$$a+b\,; \frac{y}{y-1}, \frac{x+y-1}{x}, 1\Big]. \qquad ...(3.7)$$

An interesting special case is obtained by taking $x \to 1$ in (3.7) :

$$F_3(f, d, a+b-d, a-d-e, a+b;\ y, 1)$$

$$= (1-y)^{-e}\, Fs\Big[e+d-f, d, d, a+b-d, b, a-d-e; a+b,$$

$$a+b, a+b; \frac{y}{y-1}, y, 1\Big] \qquad ...(3.8)$$

where F_3 is Appell's function (Erdélyi *et al.* 1953, p. 224).

References

Erdélyi, A., *et al.* (1953). Higher Transcendental Functions, Vol. 1. McGraw-Hill Book Co., Inc., New York.

MacRobert, T. M. (1962). Functions of a Complex Variable, 5th Ed. Macmillan & Co., London.

Pathan, M. A. (1978). On some transformations of triple hypergeometric series $F^{(3)}$—I. *Indian J. pure appl. Math.*, **9**, No. 4.

Saran, S. (1954). Hypergeometric functions of three variables. *Ganita*, **5**, 77-91; *Corr. ibid*, **7** (1956), 65.

Slater, L. J. (1960). Confluent Hypergeometric Functions. University Press, Cambridge.

Srivastava, H. M. (1967). Generalized Neumann expansions involving hypergeometric functions. *Proc. Camb. phil. Soc.*, **63**, 425-29.