ON THE LIMIT-2 CASE OF SECOND-ORDER MATRIX DIFFERENTIAL EQUATIONS

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The second-order matrix differential equation $(N - \lambda)\phi = 0$ $[0 \leqslant x < \infty)$ has been considered and sufficient conditions on the coefficients have been discussed under which the equation is in the limit-2 case at infinity.

§1. We consider the second-order matrix differential equation

$$(N - \lambda)\phi = 0, \ [0 \leqslant x < \infty) \qquad \ldots(1.1)$$

where N denotes the matrix operator

$$N \equiv \begin{pmatrix} -\dfrac{d}{dx}\left(p_0(x)\dfrac{d}{dx}\right) + p_1(x) & r(x) \\ r(x) & -\dfrac{d}{dx}\left(q_0(x)\dfrac{d}{dx}\right) + q_1(x) \end{pmatrix}$$

ϕ a vector having two components $u \equiv u(x)$ and $v \equiv v(x)$ represented as a column matrix $\phi = \begin{pmatrix} u \\ v \end{pmatrix}$ and the coefficients p_0, q_0, p_1, q_1, r are real-valued.

It has been shown in Bhagat (1969b) that the differential equation (1.1) has at least two linearly independent solutions, for each λ such that im $\lambda \neq 0$, which belong to the class $L^2\,[0, \infty)$ and may have three or all the four solutions belonging to the class $L^2\,[0, \infty)$. If for one strictly complex value of λ, the equation (1.1) has two or three or four linearly independent solutions belonging to $L^2\,[0, \infty)$, then this holds for all such values of λ. The differential equation (1.1) is said to be in the limit-2 or limit-3 or limit-4 case at infinity according as it has at most two or three or four linearly independent solutions belonging to $L^2\,[0, \infty)$. This classification is analogous to one made by Weyl for the second-order differential equation

$$(py^{(1)})^{(1)} + (\lambda - q)y = 0, \quad [0 \leqslant x < \infty). \qquad \ldots(1.2)$$

The differential equation (1.2) is said to be in the limit-point or limit-circle case at infinity according as it has at most one or two linearly independent solutions belonging to $L^2\,[0, \infty)$ (*see* Coddington and Levinson 1955, Titchmarsh 1950). For similar classification of the fourth-order differential equation

$$y^{(4)} - (py^{(1)})^{(1)} + qy = \lambda y, \qquad [0 \leqslant x < \infty) \qquad \text{...(1.3)}$$

one can see Everitt (1968).

Many authors have discussed sufficient conditions on p and q for the equation (1.2) to be in the limit-point case at infinity. We refer the readers for some such discussions to Everitt (1966), Coddington and Levinson (1955) and Titchmarsh (1950, 1962). During the past few years a number of papers have dealt with the limit-2 case of the equation (1.3) (*see* Devinatz 1973a, b ; Eastham 1971 ; Everitt 1968, 1969 ; Walker 1971).

Results of Everitt (1968) for the equation (1.3) have been extended to the equation (1.1) by Shaw and Bhagat (1974). The object of the present paper is to obtain sufficient conditions on p_0, q_0, p_1, q_1 and r for the equation (1.1) to be in the limit-2 case at infinity similar to those for the second-order equation (1.2) of Titchmarsh (1950) and fourth-order equation (1.3) of Everitt (1969). The result obtained here is more general than the one by Titchmarsh (1950).

§2. Let D denote the subset of the complex Hilbert space $L^2[0, \infty)$ of vector functions which is given by $\phi = \begin{pmatrix} U \\ V \end{pmatrix} \in D$ if

(a) $\phi \in L^2\,[0, \infty)$,

(b) $\phi^{(1)}$ is absolutely continuous on $[0, X]$ for all $X > 0$,

(c) $N\phi \in L^2\,[0, \infty)$.

Green's formula for the matrix differential expression $N\phi$ is given by (*see* Bhagat 1969a, §4)

$$\int_0^X (\phi_1^T\, N\bar{\phi}_2 - \bar{\phi}_2^T\, N\phi_1)\, dx = [\phi_1\phi_2]\,(X) - [\phi_1\phi_2]\,(0) \qquad \text{...(2.1)}$$

where the bilinear concomitant $[\phi_1\phi_2]\,(x)$ of two vectors $\phi_1(x) = \begin{pmatrix} U_1(x) \\ V_1(x) \end{pmatrix}$ and $\phi_2(x) = \begin{pmatrix} U_2(x) \\ V_2(x) \end{pmatrix}$ is defined as (*see* Bhagat 1969a, §3)

$$[\phi_1\phi_2](x) = p_0(U_1^{(1)}\,\bar{U}_2 - U_1\bar{U}_2^{(1)}) + q_0(V_1^{(1)}\,\bar{V}_2 - V_1\bar{V}_2^{(1)}). \qquad \text{...(2.2)}$$

From the Green's formula (2.1) it is obvious that $\lim_{X \to \infty} [\phi_1\phi_2]\,(X)$ exists and is finite.

Now the differential equation (1.1) is in the limit-2 case at infinity i.e. it has exactly two linearly independent solutions, for each λ such that $\operatorname{im} \lambda \neq 0$, which belong to the class $L^2 [0, \infty)$, if and only if

$$\lim_{X \to \infty} [\phi_1 \phi_2] (X) = 0 \qquad \ldots(2.3)$$

for all $\phi_1, \phi_2 \in D$ [*see* Shaw and Bhagat (1974) and references therein].

§3. We shall now prove the following theorem.

Theorem — If the real valued functions $p_0(x), q_0(x), p_1(x), q_1(x), r(x)$ satisfy on $[0, \infty)$

(i) $p_0(x), q_0(x) > 0$ for all $x \in [0, \infty)$,

(ii) $p_0(x), q_0(x)$ are absolutely continuous on $[0, X]$ for all $X > 0$,

(iii) $p_1(x), q_1(x) \in L [0, X]$ for all $X > 0$,

(iv) $p_0(x), q_0(x) \leqslant m x^\beta$ for all $x \in [0, \infty)$, where m is a constant $\in (0, \infty)$,

(v) $p_1(x), q_1(x) \geqslant -kx^\alpha$, $r(x) \geqslant -k_1 x^\gamma$, where k and k_1 are constants $\in (0, \infty)$,

(vi) $\alpha + \beta \leqslant 2$, $\alpha \geqslant 0$, $\beta - \alpha \leqslant 2$ and $\gamma \leqslant \alpha$;

then the equation (1.1) is in the limit-2 case at infinity.

Let $\phi = \begin{pmatrix} u \\ v \end{pmatrix}$ be any real-valued vector of D, and H and S be vectors given by

$$H = \begin{pmatrix} p_0^{1/2} u^{(1)} \\ q_0^{1/2} v^{(1)} \end{pmatrix} \text{ and } S = \begin{pmatrix} p_0^{1/2} u \\ q_0^{1/2} v \end{pmatrix}.$$

Then we have the identity

$$x^{-\alpha} \{H^T H + p_1 u^2 + q_1 v^2 + 2ruv\} = x^{-\alpha} \frac{d}{dx} (S^T H) + x^{-\alpha} \phi^T N \phi.$$

Integrating this over $[1, X]$ we get

$$\int_1^X x^{-\alpha} \{H^T H + p_1 u^2 + q_1 v^2 + 2ruv\}\, dx = \Big[x^{-\alpha} S^T H \Big]_1^X$$

$$+ \int_1^X x^{-\alpha} \phi^T N \phi \, dx + \alpha \int_1^X x^{-\alpha-1} S^T H \, dx.$$

Integrating twice we obtain after division by X^2

$$\frac{1}{2!}\int_1^X \left(1-\frac{x}{X}\right)^2 x^{-\alpha}\{H^TH + p_1u^2 + q_1v^2 + 2ruv\}\,dx$$

$$= X^{-1}\int_1^X \left(1-\frac{x}{X}\right) x^{-\alpha} S^TH\,dx$$

$$+ \frac{\alpha}{2!}\int_1^X \left(1-\frac{x}{X}\right)^2 x^{-\alpha-1} S^TH\,dx$$

$$+ \frac{1}{2!}\int_1^X \left(1-\frac{x}{X}\right)^2 x^{-\alpha}\phi^TN\phi\,dx + O(1).$$

Hence

$$\frac{1}{2!}\int_1^X \left(1-\frac{x}{X}\right)^2 x^{-\alpha}\{H^TH + (p_1+kx^{\alpha})u^2 + (q_1+kx^{\alpha})v^2 + 2(r+k_1x^{\gamma})uv\}dx$$

$$= X^{-1}\int_1^X \left(1-\frac{x}{X}\right) x^{-\alpha} S^TH\,dx$$

$$+ \frac{\alpha}{2!}\int_1^X \left(1-\frac{x}{X}\right)^2 x^{-\alpha-1} S^TH\,dx$$

$$+ \frac{1}{2!}\int_1^X \left(1-\frac{x}{X}\right)^2 x^{-\alpha}\phi^TN\phi\,dx$$

$$+ \frac{1}{2!}\int_1^X \left(1-\frac{x}{X}\right)^2 k\phi^T\phi\,dx$$

$$+ \int_1^X \left(1-\frac{x}{X}\right)^2 k_1 x^{\gamma-\alpha} uv\,dx + O(1)$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + O(1), \text{ (say)}. \qquad \ldots(3.1)$$

For all real $\phi_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $\phi_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ we have

$$(u_1u_2 + v_1v_2)^2 \leqslant (u_1^2 + v_1^2)\,(u_2^2 + v_2^2)$$

whence

$$\left| \int_a^b \phi_1^T \phi_2 \, dx \right| \leqslant \int_a^b (u_1^2 + v_1^2)^{1/2} (u_2^2 + v_2^2)^{1/2} \, dx$$

$$\leqslant \left\{ \int_a^b \phi_1^T \phi_1 \, dx \int_a^b \phi_2^T \phi_2 \, dx \right\}^{1/2}. \qquad \ldots(3.2)$$

From the inequality (3.2),

$$I_1 = O \left\{ \int_1^X \left(1 - \frac{x}{X}\right)^2 x^{-\alpha} H^T H \, dx \cdot X^{-2} \int_1^X x^{-\alpha} S^T S \, dx \right\}^{1/2}$$

$$= O \left\{ \int_1^X \left(1 - \frac{x}{X}\right)^2 x^{-\alpha} H^T H \, dx \cdot X^{-2} \int_1^X x^{\beta-\alpha} \phi^T \phi \, dx \right\}^{1/2}$$

$$= O \left\{ \int_1^X \left(1 - \frac{x}{X}\right)^2 x^{-\alpha} H^T H \, dx \right\}^{1/2}, \text{ by the condition (vi).}$$

$$I_2 = O \left\{ \int_1^X \left(1 - \frac{x}{X}\right)^2 x^{-\alpha} H^T H \, dx \cdot \int_1^X \left(1 - \frac{x}{X}\right)^2 x^{-\alpha-2} S^T S \, dx \right\}^{1/2}$$

$$= O \left\{ \int_1^X \left(1 - \frac{x}{X}\right)^2 x^{-\alpha} H^T H \, dx \cdot \int_1^X \left(1 - \frac{x}{X}\right)^2 x^{\beta-\alpha-2} \phi^T \phi \, dx \right\}^{1/2}$$

$$= O \left\{ \int_1^X \left(1 - \frac{x}{X}\right)^2 x^{-\alpha} H^T H \, dx \right\}^{1/2}, \text{ by the condition (vi).}$$

$$I_3 = O(1),$$

$$I_4 = O(1)$$

and

$$I_5 = O \left\{ \left(1 - \frac{x}{X}\right)^2 k_1 x^{\gamma-\alpha} \phi^T \phi \, dx \right\} = O(1).$$

Since $p_1 + kx^\alpha \geqslant 0$, $q_1 + kx^\alpha \geqslant 0$ and $r + k_1x^\gamma \geqslant 0$, the integrand on the left-hand side of (3.1) is non-negative. Thus if we denote the left-hand side of (3.1) by $F(X)$, we have

$$F(X) = O(\{F(X)\}^{1/2}) + O(1), \text{ as } X \to \infty.$$

Hence as $X \to \infty$, $F(X) = O(1)$.

Therefore

$$0 \leqslant \frac{1}{2^2} \cdot \frac{1}{2!} \int_1^{X/2} x^{-\alpha} \{H^T H + (p_1 + kx^\alpha) u^2 + (q_1 + kx^\alpha) v^2 + 2(r + k_1 x^\gamma) uv\} \, dx$$

$$\leqslant F(X) = O(1)$$

and so

$$\int_1^\infty x^{-\alpha} \{H^T H + (p_1 + kx^\alpha) u^2 + (q_1 + kx^\alpha) v^2 + 2(r + k_1 x^\gamma) uv\} \, dx$$

$$= O(1).$$

Hence, for any real $\phi \in D$,

$$x^{-(\alpha/2)} H \in L^2 [1, \infty) \qquad \ldots(3.3)$$

and

$$x^{-\alpha/2} \{(p_1 + kx^\alpha) u^2 + (q_1 + kx^\alpha) v^2 + 2(r + k_1 x^\gamma) uv\} \in L[1, \infty).$$

Also we have

$$\int_1^X x^{\alpha-2} S^T S \, dx \leqslant m \int_1^X x^{\beta+\alpha-2} \mid \phi \mid^2 dx < \infty, \text{ by (vi)}$$

so that

$$x^{(\alpha-2)/2} S \in L^2 [1, \infty). \qquad \ldots(3.4)$$

For general $\phi = \begin{pmatrix} U \\ V \end{pmatrix} \in D$ the results (3.3) and (3.4) follow on separating ϕ into real and imaginary parts.

Let us now assume on the contrary to (2.3) that there is a pair of vectors $\phi_1 = \begin{pmatrix} U_1 \\ V_1 \end{pmatrix}$ and $\phi_2 = \begin{pmatrix} U_2 \\ V_2 \end{pmatrix} \in D$ such that $\lim_{X \to \infty} [\phi_1 \phi_2](X) \neq 0$. It follows that there is a pair of real of real-valued vectors $\theta_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $\theta_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in D$ such that $\lim_{X\to\infty} [\theta_1\theta_2](X) = 2M > 0$.

So we can choose $a \geqslant 1$ such that

$$[\theta_1\theta_2](X) \geqslant M > 0 \text{ for } X \geqslant a.$$

Then from (2.2), we have

$$\int_a^X \frac{M}{x}\, dx \leqslant \frac{[\theta_1\theta_2](x)}{x}\, dx$$

$$= \int_a^X \frac{p_0 u_1^{(1)} u_2 + q_0 v_1^{(1)} v_2}{x}\, dx - \int_a^X \frac{p_0 u_1 u_2^{(1)} + q_0 v_1 v_2^{(1)}}{x}\, dx.$$

Now

$$\int_a^X \frac{p_0 u_1^{(1)} u_2 + q_0 v_1^{(1)} v_2}{x}\, dx = \int_a^X x^{-\alpha/2} H_1^T \cdot x^{(\alpha-2)/2} S_2\, dx$$

$$= O(1), \text{ by (3.3) and (3.4).} \qquad \ldots(3.5)$$

In the same way

$$\int_a^X \frac{p_0 u_1 u_2^{(1)} + q_0 v_1 v_2^{(1)}}{x}\, dx = O(1). \qquad \ldots(3.6)$$

Therefore, from (3.5) and (3.6), we have

$$\int_a^X \frac{M}{x}\, dx = O(1), \text{ as } X \to \infty$$

which is a contradiction if $M \neq 0$. This contradiction can only be removed if $\lim_{X\to\infty} [\theta_1\theta_2](X) = 0$ and so (2.3) holds for all $\phi_1, \phi_2 \in D$.

This completes the proof of the theorem.

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