AXISYMMETRIC CONTACT PROBLEM FOR A GRANULAR LAYER LYING OVER AN ELASTIC FOUNDATION*

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The title problem is equivalent to an axisymmetric mixed boundary value problem in the granular theory of elasticity, for anin finite granular layer, in which the normal displacement is prescribed inside the circular area $r \leqslant 1$, the normal stress is zero outside the circular area on the face $z = -h$ and the shearing stress is zero on the whole of this face : the continuity of the normal and radial displacements and the normal and shearing stresses is assumed at the interface $z = 0$ between the granular layer $(-h \leqslant z \leqslant 0)$ and the elastic foundation $(z \geqslant 0)$. The problem is reduced to the solution of a Fredholm integral equation of the second kind; for a flat-ended circular punch its iterative solution has been obtained for values of $h >> 1$. Simple expressions have been obtained for the total pressure P on the punch for a given depth of penetration and the normal stress under the punch.

1. INTRODUCTION

In nature the layered soils being a common occurrence, many problems of linear theory of elasticity, especially in the field of soil mechanics, have been treated as layered elastic systems by some investigators (Westergaard 1938, Burmister 1945, Acum and Fox 1951, Lemcoe 1960, Poulos 1967). The classical theory of elasticity is known to give inaccurate description of the state of granular substances like soil under load. Weiskopf (1945) proposed a linear theory which takes into account not only the elastic behaviour of such substances under compression and shear but also the slipping of the granules on each other. Since slipping reduces resistance to shear, deflection due to shear is bound to be greater for such substances than for non-granular solids, which implies that

$$\frac{E}{\mu} > 2(1 + \nu) \qquad \text{...(1)}$$

where E and μ are the modulii of elasticity in compression and shear respectively and ν is Poisson's ratio. The ratio E/μ may be regarded as a material constant. In the classical theory of elasticity $E/\mu = 2(1 + \nu)$.

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Das (1962) introduced two stress functions $\phi(r, z)$ and $\psi(r, z)$ to solve the axisymmetric equations of equilibrium for granular materials and expressed the stresses and displacements in terms of these two stress functions where ϕ and ψ must satisfy coupled system of partial differential equations and (r, θ, z) is a cylindrical polar coordinate system. Das (1962) solved the problems of a granular layer resting on a rigid foundation under Gaussian distribution of load. Recently Misra and Sen (1975; 1976a, b) solved the problems of a granular half-space and a granular layer resting over a rigid smooth or rough foundation under given axisymmetric surface loads. Ejike (1977a, b) has solved the axisymmetric contact problems for a granular half-space as well as a granular layer lying over a rigid foundation.

In this paper we consider the axisymmetric contact problem for a granular layer lying over an elastic half-space. The punch is assumed to be rigid but of arbitrary axisymmetric profile. The solutions for corresponding problems of a granular layer lying over a rigid foundation and a granular half-space considered by Ejike (1977a, b) are particular cases of this more general problem. The problem of an elastic layer lying over an elastic foundation has been considered by Dhaliwal (1970), Dhaliwal and Rau (1970) and Rau and Dhaliwal (1972). The problem is reduced to the solution of a Fredholm integral equation of the second kind, whose iterative solution has been obtained for values of layer thickness much greater than the radius of contact of the punch. For thin layers the intergal equation may be solved numerically following Dhaliwal (1970). Simple expressions have been obtained for the normal stress under the punch and the total pressure P required on the punch for a given depth of penetration. Detailed iterative solution has been obtained for a cylindrical punch. Similar iterative solutions may be obtained for conical, spherical, paraboloidal and ellipsoidal punch shapes as can be seen from Dhaliwal and Rau (1970) who have obtained solutions for all the above shapes.

2. Basic Equations for Granular Material and Their Solution

The basic equations of the linear theory of elasticity for granular materials as proposed by Weiskopf (1945) are as follows:

Equations of equilibrium in the absence of body forces :

$$\tau_{ij,j} = 0, \quad \tau_{ij} = \tau_{ji}, \quad (i, j = 1, 2, 3). \qquad \text{...(2)}$$

Stress-strain relations :

$$e_{ij} = \begin{cases} \dfrac{1}{E_1} [(1 + \nu_1)\, \tau_{ij} - \nu_1 \tau_{kk}], & i = j, \\ \dfrac{1}{2\mu_1}\, \tau_{ij}, & i \neq j, \end{cases} \quad (i, j, k = 1, 2, 3). \qquad \text{...(3)}$$

Displacement-strain relations :

$$e_{ij} = u_{i,j} + u_{j,i}, \quad (i, j = 1, 2, 3). \qquad \text{...(4)}$$

Compatibility equations:

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0, \quad (i, j, k, l = 1, 2, 3) \qquad \text{...(5)}$$

where τ_{ij} and e_{ij} are respectively the stress and the strain tensors, u_i are the displacement components and E_1, μ_1 and ν_1 are respectively the Young's modulus, shear modulus and Poisson's ratio of the material.

In cylindrical polar coordinates (r, θ, z), Das (1962) has shown that for axisymmetric case if we introduce two stress functions $\phi(r, z)$ and $\psi(r, z)$, the displacement field $\vec{u} = (u_r, u_\theta, u_z)$ is given by

$$\vec{u}(r, z) = \frac{1 + \nu_1}{E_1} \left\{ -\frac{\partial}{\partial r} (\psi + k\phi), 0, \frac{\partial}{\partial z} (\psi - \phi) \right\} \qquad \text{...(6)}$$

where

$$k = \frac{1}{1 + \nu_1} \left[\frac{E_1}{\mu_1} - (1 + \nu_1) \right] > 1 \qquad \text{...(7)}$$

and the stress components are now given by

$$\left.\begin{aligned} \sigma_{rr}(r, z) &= \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} (\psi + k\phi) \\ \sigma_{zr}(r, z) &= -\frac{\partial^2 \phi}{\partial r \partial z} \\ \sigma_{zz}(r, z) &= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \\ \sigma_{\theta\theta}(r, z) &= \nu_1 \nabla^2 \phi - \frac{1}{r} \frac{\partial}{\partial r} (\psi + k\phi) \end{aligned}\right\} \qquad \text{...(8)}$$

where ∇^2 is the Laplacian operator and ϕ and ψ must satisfy the following system of partial differential equations :

$$\nabla^4 \phi = \frac{k - 1}{1 - \nu_1} \frac{\partial^2}{\partial z^2} \left(\frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi \right) \qquad \text{...(9)}$$

$$\nabla^4 \psi = \frac{k - 1}{1 - \nu_1} \frac{\partial^2}{\partial z^2} \left(\frac{\partial^2 \psi}{\partial z^2} - \nabla^2 \psi \right) \qquad \text{...(10)}$$

$$\frac{\partial^2 \psi}{\partial z^2} = (1 - \nu_1) \nabla^2 \phi. \qquad \text{...(11)}$$

Let us denote the Hankel transform of order n of the function $f(\xi, z)$ with respect to the variable ξ by the notation :

$$H_n[f(\xi, z); \xi \to r] = \int_0^\infty \xi f(\xi, z) J_n(\xi r)\, d\xi \qquad ...(12)$$

and similarly

$$H_n[f(r, z); r \to \xi] = \int_0^\infty r f(r, z) J_n(\xi r)\, dr. \qquad ...(13)$$

Applying the Hankel transform of order zero with respect to r to eqns. (9) and (10) we find that solution for ϕ and ψ may be taken in the form ;

$$\phi(r, z) = H_0[A_1 e^{\alpha\xi z} + B_1 e^{-\alpha\xi z} + C_1 e^{\beta\xi z} + D_1 e^{-\beta\xi z}; \xi \to r] \qquad ...(14)$$

$$\psi(r, z) = H_0[\bar{A}_1 e^{\alpha\xi z} + \bar{B}_1 e^{-\alpha\xi z} + \bar{C}_1 e^{\beta\xi z} + \bar{D}_1 e^{-\beta\xi z}; \xi \to r] \qquad ...(15)$$

where α^2 and β^2 are the roots of the equation

$$x^2 + (k' - 2)x + 1 = 0; \; k' = \frac{(1-k)}{(1-\mu_1)} \qquad ...(16)$$

and A_1, B_1, C_1, D_1, $\bar{A}_1$, $\bar{B}_1$, $\bar{C}_1$ and $\bar{D}_1$ are functions of ξ.

From eqns. (11), (14) and (15), we find that

$$\bar{A}_1 = LA, \; \bar{B}_1 = LB, \; \bar{C}_1 = MC, \; \bar{D}_1 = MD \qquad ...(17)$$

where

$$L = \frac{(k-1)}{(\alpha^2 - 1)}, \; M = -\alpha^2 L. \qquad ...(18)$$

Inserting the expressions for ϕ and ψ from eqns. (14) and (15) into eqns. (6) and (8) and utilizing the relations (17), we find the following expressions for the displacement and stress components :

$$E_1 u_r(r, z) = (1 + \nu_1) H_1 [(L + k)(A_1 e^{\alpha\xi z} + B_1 e^{-\alpha\xi z})\xi + (M + k)(C_1 e^{\beta\xi z} + D_1 e^{-\beta\xi z})\xi; \xi \to r] \qquad ...(19)$$

$$E_1 u_z(r, z) = (1 + \nu_1) H_0 [\alpha(L - 1)(A_1 e^{\alpha\xi z} - B_1 e^{-\alpha\xi z})\xi + \beta(M - 1)(C_1 e^{\beta\xi z} - D_1 e^{-\beta\xi z})\xi; \xi \to r] \qquad ...(20)$$

$$\sigma_{zz}(r, z) = -H_0[(A_1 e^{\alpha\xi z} + B_1 e^{-\alpha\xi z} + C_1 e^{\beta\xi z} + D_1 e^{-\beta\xi z})\xi^2; \xi \to r] \qquad ...(21)$$

$$\sigma_{rz}(r, z) = H_1[(A_1 e^{\alpha\xi z} - B_1 e^{-\alpha\xi z})\alpha\xi^2 + (C_1 e^{\beta\xi z} - D_1 e^{-\beta\xi z})\beta\xi^2; \xi \to r]. \qquad ...(22)$$

3. Basic Equations for Elastic Material and Their Solution

In the case of axial symmetry, the displacement vector $\vec{u}$ assumes the form $(u_r, 0, u_z)$ in a cylindrical coordinate system (r, θ, z). The equations of equilibrium and the required stress-displacement relations for a homogeneous, isotropic, elastic medium are given by

$$\mu_2 \nabla^2 \vec{u} + (\lambda_2 + \mu_2) \nabla(\nabla \cdot \vec{u}) = 0 \qquad ...(23)$$

$$\left.\begin{aligned} \sigma_{zz}(r, z) &= (\lambda_2 + 2\mu_2) \frac{\partial u_z}{\partial z} + \lambda_2 \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} \right) \\ \sigma_{rz}(r, z) &= \mu_2 \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \end{aligned}\right\} \qquad ...(24)$$

where λ_2, μ_2 are the Lamé constants.

The solution of the system of partial differential eqns. (23) is given by

$$2\mu_2 u_r(r, z) = \frac{\partial \chi}{\partial r} + z \frac{\partial \Omega}{\partial r} \qquad ...(25)$$

$$2\mu_2 u_z(r, z) = \frac{\partial \chi}{\partial z} + z \frac{\partial \Omega}{\partial z} - (3 - 4\nu_2)\, \Omega \qquad ...(26)$$

where $\chi(r, z)$ and $\Omega(r, z)$ are harmonic functions and ν_2 is Poisson's ratio.

We take the following representation for these harmonic functions :

$$\left.\begin{aligned} \chi(r, z) &= H_0 \left[A_2(\xi)\, e^{-\xi z} + B_2(\xi)\, e^{\xi z};\, \xi \to r\right] \\ \Omega(r, z) &= H_0 \left[C_2(\xi)\, e^{-\xi z} + D_2(\xi)\, e^{\xi z};\, \xi \to r\right] \end{aligned}\right\} \qquad ...(27)$$

where the functions A_2, B_2, C_2 and D_2 are arbitrary functions of ξ.

Inserting the expressions for χ and Ω from (27) into (26), and again using these in (24) we obtain the following expressions for the displacement and stress components :

$$2\mu_2 u_r(r, z) = -H_1 \left[(A_2 + zC_2)\, \xi e^{-\xi z} + (B_2 + zD_2)\, \xi e^{\xi z};\, \xi \to r\right] \qquad ...(28)$$

$$2\mu_2 u_z(r, z) = H_0 \left[-\{\xi(A_2 + zC_2) + (3 - 4\nu_2)\, C_2\}\, e^{-\xi z} + \{\xi(B_2 + zD_2) - (3 - 4\nu_2)\, D_2\}\, e^{\xi z};\, \xi \to r\right] \qquad ...(29)$$

$$\sigma_{zz}(r, z) = H_0 \left[\{\xi(A_2 + zC_2) + 2(1 - \nu_2)\, C_2\}\, \xi e^{-\xi z} + \{\xi(B_2 + zD_2) - 2(1 - \nu_2)\, D_2\}\, \xi e^{\xi z};\, \xi \to r\right] \qquad ...(30)$$

$$\sigma_{rz}(r, z) = -H_1 \left[-\{\xi(A_2 + zC_2) + (1 - 2\nu_2)\, C_2\}\, \xi e^{-\xi z} + \{\xi(B_2 + zD_2) - (1 - 2\nu_2)\, D_2\}\, \xi e^{\xi z};\, \xi \to r\right]. \qquad ...(31)$$

4. Statement of the Problem and Its Solution

We consider an infinite, isotropic, homogeneous, elastic granular layer bounded by the planes $z = 0$ and $z = -h$ and in perfect contact with the semi-infinite isotropic, homogeneous, elastic foundation ($z \geqslant 0$). The free surface $z = -h$ is indented over a circular area by a rigid punch of arbitrary profile. The stress and displacement fields in the granular layer are given by eqns. (19) to (22) and the stress and displacement fields in the elastic half-space are given by eqns. (28) to (31) by taking $B_2 = D_2 = 0$ so that the stresses and displacements tend to zero as $z \to \infty$. Our problem now is to determine the solution satisfying the boundary conditions

$$u_z^{(1)}(r, -h) = \delta - f(r), \quad 0 \leqslant r \leqslant 1 \qquad \ldots(32)$$

$$\sigma_{zz}^{(1)}(r, -h) = 0, \quad r > 1 \qquad \ldots(33)$$

$$\sigma_{rz}^{(1)}(r, -h) = 0, \quad r \geqslant 0 \qquad \ldots(34)$$

where the function $f(r)$ is prescribed by the fact that, referred to the tip as origin, the punch has the equation $z = f(r)$ so that $f(0) = 0$; the radius of the circle of contact is unity and δ is a parameter (as yet unspecified) whose physical significance is that it is the depth to which the tip of the punch penetrates the elastic layer. The superscripts 1 and 2 refer respectively to the regions 1 ($-h \leqslant z \leqslant 0$) and 2 ($z \geqslant 0$).

The granular layer and the elastic foundation being in perfect contact, the following continuity conditions must be satisfied at the interface $z = 0$:

$$\left.\begin{aligned} [u_z^{(i)}(r, 0)]_1^2 = [u_r^{(i)}(r, 0)]_1^2 = 0 \\ [\sigma_{zz}^{(i)}(r, 0)]_1^2 = [\sigma_{rz}^{(i)}(r, 0)]_1^2 = 0 \end{aligned}\right\} r \geqslant 0. \qquad \ldots(35)$$

Utilizing eqns. (19) - (22) and (28) - (31), we find that the continuity conditions (35) will be satisfied if

$$\left.\begin{aligned} C_1 &= \xi^{-1}A_2(\rho_0 + \beta^{-1}\rho_1\xi) + \xi^{-1}C_2(\rho_2 + \rho_3\xi^{-1} + \rho_1\xi) \\ D_1 &= \xi^{-1}A_2(\rho_4 - \beta^{-1}\rho_1\xi) + \xi^{-1}C_2[-\rho_2 + (\rho_5 - \rho_3)\xi^{-1} + \rho_1\xi] \\ A_1 &= \frac{1-\alpha}{2\alpha}\xi^{-1}A_2 + \rho_6\xi^{-2}C_2 - \frac{\alpha+\beta}{2\alpha}C_1 - \frac{\alpha-\beta}{2\alpha}D_1 \\ B_1 &= -\frac{1+\alpha}{2\alpha}\xi^{-1}A_2 + \rho_7\xi^{-2}C_2 - \frac{\alpha-\beta}{2\alpha}C_1 - \frac{\alpha+\beta}{2\alpha}D_1 \end{aligned}\right\} \qquad \ldots(36)$$

where

$$\rho_0 = [\beta^{-1}(L - 1) - (L + k)]\,[2(L - M)]^{-1},$$

$$\rho_1 = E_1\,[4\mu_2(1 + \nu_1)\,(L - M)]^{-1},$$

(*continued on p.* 674)

$$\rho_2 = E_1(3 - 4\nu_2)\,[4\mu_2\beta(1 + \nu_1)\,(L - M)]^{-1},$$

$$\rho_3 = [\beta^{-1}(1 - 2\nu_2)\,(L - 1) - 2(1 - \nu_2)\,(L + k)]\,[2(L - M)]^{-1},$$

$$\rho_4 = -\,\rho_0 - (L + k)\,(L - M)^{-1},$$

$$\rho_5 = -\,2(1 - \nu_2)\,(L + k)\,(L - M)^{-1},$$

$$\rho_6 = [(1 - 2\nu_2)\,(2\alpha)^{-1} - (1 - \nu_2)],$$

$$\rho_7 = -\,[1 - \nu_2 + (1 - 2\nu_2)\,(2\alpha)^{-1}]. \qquad \ldots(37)$$

Thus eqns. (36) express the functions A_1, B_1, C_1 and D_1 in terms of two functions A_2 and C_2. The boundary condition (34) gives the following relationship between A_2 and C_2:

$$A_2(\xi) = \frac{X(\xi, h)\, C_2(\xi)}{Y(\xi, h)} \qquad \ldots(38)$$

where

$$X(\xi, h) = \alpha\,[\rho_6 \exp(-\,\alpha\xi h) - \rho_7 \exp(\alpha\xi h)] + (\rho_3 + \rho_2\xi + \rho_1\xi^2)\,X_1(\xi h) + (\rho_5 - \rho_3 - \rho_2\xi + \rho_1\xi^2)\,X_2(\xi h),$$

$$Y(\xi, h) = -\,\xi\,[\cosh(\alpha\xi h) + \alpha \sinh(\alpha\xi h) + \beta^{-1}\{(\beta\rho_0 + \rho_1\xi)\,X_1(\xi h) + (\beta\rho_4 - \rho_1\xi)\,X_2(\xi h)\}], \qquad \ldots(39)$$

with

$$X_1(x) = \alpha \sinh(\alpha x) - \beta \cosh(\alpha x) + \beta \cosh(\beta x) - \beta \sinh(\beta x),$$

$$X_2(x) = \alpha \sinh(\alpha x) + \beta \cosh(\alpha x) - \beta \cosh(\beta x) - \beta \sinh(\beta x). \qquad \ldots(40)$$

Now substituting for A_1, B_1, C_1, D_1 and A_2 in terms of C_2 from eqns. (36) – (40) in eqns. (20) and (21), we find that

$$\sigma_{zz}^{(1)}(r, -\,h) = -\,H_0\,[\xi^2 G(\xi);\, \xi \to r] \qquad \ldots(41)$$

$$E_1 u_z^{(1)}(r, -\,h) = (1 + \nu_1)\,H_0\,[\xi\{1 + H(\xi, h)\}\,G(\xi);\, \xi \to r] \qquad \ldots(42)$$

where

$$G(\xi)\,Y(\xi, h) = C_2(\xi)\,\{[\{(\beta\rho_0 + \rho_1\xi)\,G_1(\xi h) - (\beta\rho_4 - \rho_1\xi)\,G_2(\xi h)\}\,(\alpha\beta)^{-1} - \exp(\alpha\xi h)]\,X(\xi, h) + [(\rho_3 + \rho_2\xi + \rho_1\xi^2)\,G_1(\xi h) + \alpha\{\rho_6 \exp(-\,\alpha\xi h) + \rho_7 \exp(\alpha\xi h)\} + (\rho_5 - \rho_3 - \rho_2\xi + \rho_1\xi^2)\,G_2(\xi h)]\,(\alpha\xi)^{-1}\,Y(\xi, h)\} \qquad \ldots(43)$$

$$H(\xi, h) = [N(\xi, h)/D(\xi, h)] - 1 \qquad \ldots(44)$$

with

$$\left.\begin{aligned} G_1(x) &= \beta \sinh(\alpha x) - \alpha \cosh(\alpha x) + \alpha \cosh(\beta x) - \alpha \sinh(\beta x) \\ G_2(x) &= \beta \sinh(\alpha x) + \alpha \cosh(\alpha x) - \alpha \cosh(\beta x) - \alpha \sinh(\beta x) \end{aligned}\right\} \quad \ldots(45)$$

$$\begin{aligned} N(\xi, h) = {} & X(\xi, h)\{(1-\alpha)(L-1)\cosh(\alpha\xi h) + \beta^{-1}(\beta\rho_0 + \rho_1\xi)[(L-1)X_3(\xi h) \\ & + \beta(M-1)\exp(-\beta\xi h)] + \beta^{-1}(\beta\rho_4 - \rho_1\xi)[(L-1)X_4(\xi h) \\ & - \beta(M-1)\exp(\beta\xi h)]\} + \xi^{-1}Y(\xi, h)\{\alpha(L-1)(\rho_6 + \rho_7) \\ & + [(L-1)X_3(\xi h) + \beta(M-1)\exp(-\beta\xi h)](\rho_3 + \rho_2\xi + \rho_1\xi^2) \\ & + (\rho_5 - \rho_3 - \rho_2\xi + \rho_1\xi^2)[(L-1)X_4(\xi h) - \beta(M-1)\exp(\beta\xi h)]\}, \end{aligned}$$

$$D(\xi, h) = \frac{G(\xi)\, Y(\xi, h)}{C_2(\xi)} \quad \ldots(46)$$

$$X_3(x) = \alpha \sinh(\alpha x) - \beta \cosh(\beta x), \quad X_4(x) = \alpha \sinh(\alpha x) + \beta \cosh(\beta x);$$

in fact the expression for $D(\xi, h)$ is given by eqn. (43).

The boundary conditions (32) and (33) are satisfied if $G(\xi)$ is the solution of the dual integral equations

$$H_0[\xi^2 G(\xi); \xi \to r] = 0, \quad r > 1 \quad \ldots(47)$$

$$H_0[\xi\{1 + H(\xi, h)\}\, G(\xi); \xi \to r] = \frac{E_1}{1+\nu_1}\{\delta - f(r)\}, \quad 0 \leqslant r < 1. \quad \ldots(48)$$

If we represent $G(\xi)$ by the relation

$$\xi^2 G(\xi) = \int_0^1 \Phi(t) \cos(t\xi)\, dt \quad \ldots(49)$$

then eqn. (47) is identically satisfied and eqn. (48) is also satisfied if $\Phi(t)$ satisfies the following Fredholm integral equation of the second kind :

$$\Phi(t) + \int_0^1 K(u, t)\, \Phi(u)\, du = \frac{2E_1}{\pi(1+\nu_1)} \frac{d}{dt} \int_0^t \frac{r\{\delta - f(r)\}}{(t^2 - r^2)^{1/2}}\, dr, \quad 0 < t < 1 \quad \ldots(50)$$

where

$$K(u, t) = \frac{2}{\pi} \int_0^\infty H(\xi, h) \cos(t\xi) \cos(u\xi)\, d\xi. \quad \ldots(51)$$

From eqns. (41) and (49), we find that the normal stress under the punch is given by

$$\sigma_{zz}^{(1)}(r, -h) = \frac{1}{r}\frac{d}{dr}\int_r^1 \frac{t\Phi(t)}{(t^2 - r^2)^{1/2}}\, dt, \quad 0 \leqslant r < 1$$

$$= -\frac{\Phi(1)}{(1 - r^2)^{1/2}} + \int_r^1 \frac{\Phi'(t)}{(t^2 - r^2)^{1/2}}\, dt, \quad 0 \leqslant r < 1 \qquad ...(52)$$

where prime denotes the derivative with respect to t. The total pressure P which must be applied to the punch to maintain the penetration δ is given by

$$P = -2\pi \int_0^1 r\sigma_{zz}^{(1)}(r, -h)\, dr. \qquad ...(53)$$

From (52) and (53) we obtain

$$P = 2\pi \int_0^1 \Phi(t)\, dt. \qquad ...(54)$$

For a smooth punch $\sigma_{zz}^{(1)}(r, -h)$ must be finite as $r \to 1-$, which implies that

$$\Phi(1) = 0. \qquad ...(55)$$

Equation (55) gives the formula for δ, the depth of penetration of the punch relative to the radius of contact, which is taken to be unity.

It can be shown that the kernel $K(u, t)$ as given by eqn. (51) is convergent, since

$$\lim_{\xi \to 0} H(\xi, h) = \text{finite quantity},$$

$$\lim_{\xi \to \infty} H(\xi, h) = 0.$$

5. Iterative Solution

To obtain an iterative solution of the integral eqn. (50) for values of $h \gg 1$, we rewrite $K(u, t)$ in the form :

$$K(u, t) = \frac{2}{\pi h}\int_0^\infty H_1(\zeta, h) \cos\left(\frac{t\zeta}{h}\right)\cos\left(\frac{u\zeta}{h}\right) d\zeta, \qquad ...(56)$$

where

$$\zeta = \xi h, \quad H(\xi, h) \equiv H_1(\zeta, h). \qquad ...(57)$$

Expanding the integrand of (56) in inverse powers of h, we may write

$$H_1(\zeta, h) = -1 + \left[f_0 + \frac{f_1}{h} + \frac{f_2}{h^2} + \frac{f_3}{h^3}\right]\left[g_0 + \frac{g_1}{h} + \frac{g_2}{h^2} + \frac{g_3}{h^3}\right]^{-1}$$

$$= \sum_{n=0}^{\infty} \frac{k_n(\zeta)}{h^2} \qquad \text{...(58)}$$

$$2\cos\left(\frac{t\zeta}{h}\right)\cos\left(\frac{u\zeta}{h}\right) = \sum_{m=0}^{\infty} \frac{(-1)^{m-1}}{(2m)!}\left(\frac{\zeta}{h}\right)^{2m} K_m(u, t) \qquad \text{...(59)}$$

where

$$K_m(u, t) = -\left[(u+t)^{2m} + (u-t)^{2m}\right], \quad m = 0, 1, 2, 3, \ldots \qquad \text{...(60)}$$

$$k_0(\zeta) = -1 + \bar{f}_0,$$

$$k_1(\zeta) = -\bar{f}_0\bar{g}_1 + \bar{f}_1,$$

$$k_2(\zeta) = -\bar{f}_0(\bar{g}_2 - \bar{g}_1^2) - \bar{f}_1\bar{g}_1 + \bar{f}_2,$$

$$k_3(\zeta) = -\bar{f}_0(\bar{g}_3 - 2\bar{g}_1\bar{g}_2 + \bar{g}_1^3) - \bar{f}_1(\bar{g}_2 - \bar{g}_1^2) - \bar{f}_2\bar{g}_1 + \bar{f}_3,$$

$$k_4(\zeta) = \bar{f}_0(2\bar{g}_1\bar{g}_3 + \bar{g}_2^2 - 3\bar{g}_1^2\bar{g}_2 + \bar{g}_1^4) - \bar{f}_1(\bar{g}_3 - 2\bar{g}_1\bar{g}_2 + \bar{g}_1^3)$$
$$- \bar{f}_2(\bar{g}_2 - \bar{g}_1^2) - \bar{f}_3\bar{g}_1,$$

$$k_5(\zeta) = \bar{f}_0(2\bar{g}_2\bar{g}_3 - 3\bar{g}_1^2\bar{g}_3 - 3\bar{g}_1\bar{g}_2^2 + 4\bar{g}_1^3\bar{g}_2 - \bar{g}_1^5)$$
$$+ \bar{f}_1(2\bar{g}_1\bar{g}_3 + \bar{g}_2^2 + 3\bar{g}_1^2\bar{g}_2 + \bar{g}_1^4)$$
$$- \bar{f}_2(\bar{g}_3 - 2\bar{g}_1\bar{g}_2 + \bar{g}_1^3) - \bar{f}_3(\bar{g}_2 - \bar{g}_1^2),$$

$$k_6(\zeta) = \bar{f}_0(\bar{g}_3^2 + \bar{g}_2^3 + 6\bar{g}_1\bar{g}_2\bar{g}_3 + 4\bar{g}_1^3\bar{g}_3 + 6\bar{g}_1^2\bar{g}_2^2)$$
$$+ \bar{f}_1(2\bar{g}_2\bar{g}_3 - 3\bar{g}_1^2\bar{g}_3 - 3\bar{g}_1\bar{g}_2^2 + 4\bar{g}_1^3\bar{g}_2 - \bar{g}_1^5)$$
$$+ \bar{f}_2(2\bar{g}_1\bar{g}_3 + \bar{g}_2^2 - 3\bar{g}_1^2\bar{g}_2 + \bar{g}_1^4) - \bar{f}_3(\bar{g}_3 - 2\bar{g}_1\bar{g}_2 + \bar{g}_1^3) \qquad \text{...(61)}$$

with

$$\bar{f}_n(\zeta) = \frac{f_n(\zeta)}{g_0(\zeta)}, \quad \bar{g}_n(\zeta) = \frac{g_n(\zeta)}{g_0(\zeta)} \qquad \text{...(62)}$$

$$f_0(\zeta) = a_0b_0 - c_0d_0,$$
$$f_1(\zeta) = a_1b_0 + a_0b_1 - (c_1d_0 + c_0d_1),$$
$$f_2(\zeta) = a_2b_0 + a_1b_1 - (c_1d_1 + c_0d_2),$$

(*continued on p. 678*)

$$f_3(\zeta) = a_2 b_1 - c_1 d_2,$$

$$g_0(\zeta) = a_0 e_0 - c_0 n_0,$$

$$g_1(\zeta) = a_1 e_0 + a_0 e_1 - (c_1 n_0 + c_0 n_1),$$

$$g_2(\zeta) = a_1 e_1 + a_2 e_0 - (c_1 n_1 + c_0 n_2),$$

$$g_3(\zeta) = a_2 e_1 - c_1 n_2, \qquad \ldots(63)$$

and

$$a_0(\zeta) = \alpha\,[\rho_6 \exp(-\alpha\zeta) - \rho_7 \exp(\alpha\zeta)] + \rho_3 X_1(\zeta) + (\rho_5 - \rho_3)\,X_2(\zeta),$$

$$b_0(\zeta) = (1-\alpha)(L-1)\cosh(\alpha\zeta) + \rho_0\,[(L-1)\,X_3(\zeta) + \beta(M-1)\exp(-\beta\zeta)] + \rho_4\,[(L-1)\,X_4(\zeta) - \beta(M-1)\exp(\beta\zeta)],$$

$$c_0(\zeta) = \cosh(\alpha\zeta) + \alpha\sinh(\alpha\zeta) + \rho_0 X_1(\zeta) + \rho_4 X_2(\zeta),$$

$$d_0(\zeta) = \alpha(L-1)(\rho_6 + \rho_7) + \rho_3\,[(L-1)\,X_3(\zeta) + \beta(M-1)\exp(-\beta\zeta)] + (\rho_5 - \rho_3)\,[(L-1)\,X_4(\zeta) - \beta(M-1)\exp(\beta\zeta)],$$

$$e_0(\zeta) = -\exp(\alpha\zeta) - \frac{1}{\alpha}\,[\rho_0 G_1(\zeta) - \rho_4 G_2(\zeta)],$$

$$n_0(\zeta) = [\rho_6 \exp(-\alpha\zeta) - \rho_7 \exp(\alpha\zeta)] + \frac{\rho_3}{\alpha}\,G_1(\zeta) + \frac{(\rho_5 - \rho_3)}{\alpha}\,G_2(\zeta),$$

$$a_1(\zeta) = \frac{\rho_2\beta}{\rho_1}\,c_1(\zeta) = \rho_2\zeta\,[X_1(\zeta) - X_2(\zeta)],$$

$$b_1(\zeta) = -2\rho_1\zeta\,[(L-1)\cosh(\beta\zeta) + (M-1)\sinh(\beta\zeta)],$$

$$d_1(\zeta) = 2\rho_2\beta\zeta(M-L)\cosh(\beta\zeta),$$

$$e_1(\zeta) = \frac{1}{\beta\zeta}\,n_2(\zeta) = \frac{\rho_1}{\alpha\beta}\,\zeta\,[G_1(\zeta) + G_2(\zeta)],$$

$$n_1(\zeta) = \frac{\rho_2}{\alpha}\,\zeta\,[G_1(\zeta) - G_2(\zeta)],$$

$$a_2(\zeta) = \rho_1\zeta^2\,[X_1(\zeta) + X_2(\zeta)],$$

$$d_2(\zeta) = 2\rho_1\zeta^2\,[(L-1)\,\alpha\sinh(\alpha\zeta) - \beta(M-1)\sinh(\beta\zeta)]. \qquad \ldots(64)$$

From eqns. (56), (58) and (59), we may now write the kernel $K(u, t)$ in the form

$$K(u, t) = \sum_{n=1}^{\infty} \frac{F_n(u, t)}{h^n} \qquad \ldots(65)$$

where

$$F_n(u, t) = \sum_{m=0}^{[(n-1)/2]} K_m(u, t)\, I_{n-2m-1, m} \qquad \text{...(66)}$$

with

$$I_{n,m} = \frac{(-1)^{m-1}}{\pi(2m)!} \int_0^\infty \zeta^{2m} k_n(\zeta)\, d\zeta;$$

$\left[\frac{n}{2}\right]$ denotes the integral part of $\frac{n}{2}$.

Let us assume an iterative solution of the integral eqn. (50) in the form :

$$\Phi(t) = \sum_{n=0}^{\infty} \frac{\Phi_n(t)}{h^n}. \qquad \text{...(67)}$$

Now substituting for $\Phi(t)$ and $K(u, t)$ respectively from eqns. (67) and (65) into eqn. (50) and comparing the coefficients of powers of h from both sides, we obtain

$$\Phi_0(t) = F(t) \qquad \text{...(68)}$$

$$\Phi_m(t) = -\sum_{n=1}^{m} \int_0^1 F_n(u, t)\, \Phi_{m-n}(u)\, du \qquad \text{...(69)}$$

where

$$F(t) = \frac{2E_1}{\pi(1 + \nu_1)} \frac{d}{dt} \int_0^t \frac{r\{\delta - f(r)\}}{(t^2 - r^2)^{1/2}}\, dr. \qquad \text{...(70)}$$

When the shape of the punch is given (i.e. $f(r)$ is prescribed), $\Phi(t)$ can be calculated from eqns. (67) - (70). The normal stress under the punch and the total pressure P on the punch may then be calculated from eqns. (52) and (54) respectively. For the case of a cylindrical punch $f(r) = 0$ and δ, the depth of penetration of the punch, is a given quantity. In this case we find

$$F(t) = p, \quad p = \frac{2E_1\delta}{\pi(1 + \nu_1)}. \qquad \text{...(71)}$$

From the form of $K_n(u, t)$ given by eqn. (60) and from eqns. (67), (68), (69) and (71), it is obvious that $\Phi(t)$ will be of the following form :

$$\Phi(t) = 2p \sum_{n=1}^{\infty} [p_{2n-1}\, t^{2n-2} + O(h^{-2n})], \quad 0 < t < 1 \qquad \text{...(72)}$$

and hence from (52) and (53), we find that

$$\sigma_{zz}^{(1)}(r, -h) = \frac{2p}{(1-r^2)^{1/2}} \sum_{n=1}^{\infty} [p_{2n-1} + O(h^{-2n})]$$

$$+ 2p \sum_{n=2}^{\infty} [(n-1)\, p_{2n-1} P_{n-1}(r) + O(h^{-2n})], \quad 0 \leqslant r < 1 \qquad \ldots(73)$$

$$P = 4\pi p \sum_{n=1}^{\infty} \left[\frac{p_{2n-1}}{2n-1} + O(h^{-2n}) \right] \qquad \ldots(74)$$

where

$$P_{n+1} = 2 \int_r^1 \frac{t^{2n+1}\, dt}{(t^2 - r^2)^{1/2}} = 2\sqrt{\pi} \sum_{i=0}^{n} {}^nC_i (-1)^i \frac{\Gamma(i+1)}{\Gamma(i+\frac{3}{2})} (1-r)^{(2i+1)/2},$$

$$n = 0, 1, 2, \ldots \qquad \ldots(75)$$

By carrying out the iteration process up to Φ_7, we find that

$$p_1 = \tfrac{1}{2} + \frac{l_1}{h} + \frac{l_2}{h^2} + \frac{l_3}{h^3} + \frac{l_4}{h^4} + \frac{l_5}{h^5} + \frac{l_6}{h^6} + \frac{l_7}{h^7},$$

$$p_3 = \frac{m_3}{h^3} + \frac{m_4}{h^4} + \frac{m_5}{h^5} + \frac{m_6}{h^6} + \frac{m_7}{h^7},$$

$$p_5 = \frac{n_5}{h^5} + \frac{n_6}{h^6} + \frac{n_7}{h^7},$$

$$p_7 = \frac{q_7}{h^7} \qquad \ldots(76)$$

where

$$l_1 = I_{0,0}, \quad l_2 = I_{1,0} + 2I_{0,0}^2,$$

$$l_3 = I_{2,0} + \tfrac{1}{3} I_{0,1} + 2l_1 I_{1,0} + 2l_2 I_{0,0}, \quad m_3 = I_{0,1}, \quad m_4 = I_{1,1},$$

$$l_4 = I_{3,0} + \tfrac{1}{3} I_{1,1} + 2l_1(I_{2,0} + \tfrac{1}{3} I_{0,1}) + 2l_2 I_{1,0} + 2(l_3 + \tfrac{1}{3} m_3) I_{0,0},$$

$$l_5 = I_{4,0} + \tfrac{1}{3} I_{2,1} + \tfrac{1}{5} I_{0,2} + 2l_1(I_{3,0} + \tfrac{1}{3} I_{1,1}) + 2l_2(I_{2,0} + \tfrac{1}{3} I_{0,1})$$
$$+ 2(l_3 + \tfrac{1}{3} m_3) I_{1,0} + 2(l_4 + \tfrac{1}{3} m_4) I_{0,0},$$

$$m_5 = I_{2,1} + 2I_{0,2} + 2l_1 I_{1,1} + 2l_2 I_{0,1}, \quad n_5 = I_{0,2}, \quad n_6 = I_{1,2} + 2l_1 I_{0,2},$$

$$l_6 = I_{5,0} + \tfrac{1}{3} I_{3,1} + \tfrac{1}{5} I_{1,2} + 2l_1(I_{4,0} + \tfrac{1}{3} I_{2,1} + \tfrac{1}{5} I_{0,2}) + 2l_2(I_{3,0} + \tfrac{1}{3} I_{1,1})$$
$$+ 2l_3(I_{2,0} + \tfrac{1}{3} I_{0,1}) + 2m_3(\tfrac{1}{3} I_{2,0} + \tfrac{1}{5} I_{0,1}) + 2(l_4 + \tfrac{1}{3} m_4) I_{1,0}$$
$$+ 2(l_5 + \tfrac{1}{3} m_5 + \tfrac{1}{5} n_5) I_{0,0},$$

(continued on p. 681)

$$m_6 = I_{3,1} + 2I_{1,2} + 2l_1(I_{2,1} + 2I_{0,2}) + 2l_2I_{1,1} + 2(l_3 + \tfrac{1}{3}m_3)\,I_{0,1},$$

$$l_7 = I_{6,0} + \tfrac{1}{3}I_{4,1} + \tfrac{1}{5}I_{2,2} + \tfrac{1}{7}I_{0,3} + 2l_1(I_{5,0} + \tfrac{1}{3}I_{3,1} + \tfrac{1}{5}I_{1,2})$$
$$+ 2l_2(I_{4,0} + \tfrac{1}{3}I_{2,1} + \tfrac{1}{5}I_{0,2}) + 2l_3(I_{3,0} + \tfrac{1}{3}I_{1,1})$$
$$+ 2m_3(\tfrac{1}{3}I_{3,0} + \tfrac{1}{5}I_{1,1}) + 2l_4(I_{2,0} + \tfrac{1}{3}I_{0,1}) + 2m_4(\tfrac{1}{3}I_{2,0} + \tfrac{1}{5}I_{0,1})$$
$$+ 2(l_5 + \tfrac{1}{3}m_5 + \tfrac{1}{5}n_5)\,I_{1,0} + 2(l_6 + \tfrac{1}{3}m_6 + \tfrac{1}{5}n_6)\,I_{0,0},$$

$$m_7 = I_{4,1} + 2I_{2,2} + 3I_{0,3} + 2l_1(I_{3,1} + 2I_{1,2}) + 2l_2(I_{2,1} + 2I_{0,2})$$
$$+ 2(l_3 + \tfrac{1}{3}m_3)\,I_{1,1} + 2(l_4 + \tfrac{1}{3}m_4)\,I_{0,1},$$

$$n_7 = I_{2,2} + 5I_{0,3} + 2l_1I_{1,2} + 2l_2I_{0,2}, \quad q_7 = I_{0,3}. \qquad \ldots(77)$$

To obtain the solution for a granular half-space we let $h \to \infty$ and find that

$$\Phi(t) = \Phi_0(t) = F(t) \qquad \ldots(78)$$

which agrees with the result obtained by Ejike (1977b). Similarly by letting $\mu_2 \to \infty$, we may obtain the results for the problem of a granular layer lying over a rigid foundation.

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