THEORY OF LIE DERIVATIVES AND MOTIONS IN SPECIAL KAWAGUCHI SPACES AND ITS SUBSPACES

by U. P. SINGH *and* S. K. D. DUBEY, *Department of Mathematics, University of Gorakhpur, Gorakhpur* 273001

(*Received* 23 *November* 1976)

In this paper, we have considered the motion, Killing equations and affine motion in special Kawaguchi spaces and its subspaces of order two. This investigation has been carried after finding the Lie derivatives of the fundamental tensor and the connection parameter of the special Kawaguchi space. The necessary and sufficient conditions under which an affine motion of an embedding space be an affine motion in the subspace have been obtained.

1. INTRODUCTION

In the special Kawaguchi space $K^{(2)}_n$ of order 2, the arc length of a curve $x^i = x^i(t)$* is given by the integral (Kawaguchi 1938)

$$s = \int [F(x, x', x'')]^{1/p}\, dt,\ p \neq 0,\ \tfrac{3}{2} \qquad \ldots(1.1)$$

where

$$F(x, x', x'') = A_i(x, x')\, x''^i + B(x, x'),\ x'^i = dx^i/dt,\ x''^i = d^2x^i/dt^2,$$

A_i, B being differentiable functions of x^i and x'^i.

From the Zermelo's conditions that the arc length in the space should remain unaltered by any transformation of the parameter t, we must have

$$A_i x'^i = 0,\ A_{i(j)} x'^j\dagger = (p-2)\, A_i,\ B_{(i)} x'^i = pB. \qquad \ldots(1.2)$$

We consider an m-dimensional subspace $K^{(2)}_m$ of $K^{(2)}_n$ represented by the equations $x^i = x^i(u^\alpha)$ and the matrix of the projection factors $p^i_\alpha = \partial_\alpha x^i$ has rank m. The fundamental tensors G_{ij} and $G_{\alpha\beta}$ of the special Kawaguchi space $K^{(2)}_n$ and its subspace $K^{(2)}_m$ are defined as follows (Yoshida 1967)

*Latin indices run from 1 to n and Greek ones $\alpha, \beta, \gamma, \delta, \ldots$ from 1 to m.

†Throughout this paper, the symbols (j) and $((j))$ denote the partial differentiation with respect to x'^j and x''^j (or u' and u'') respectively and $\partial_j = \partial/\partial x^j$, $\partial_\alpha = \partial/\partial u^\alpha$.

$$G_{ij} \overset{def}{=} 2A_{i(j)} - A_{j(i)},\ G_{\alpha\beta} \overset{def}{=} 2a_{\alpha(\beta)} - a_{\beta(\alpha)} \qquad ...(1.3)$$

and we have the relations

$$G_{ij}\, p^i_\alpha\, p^j_\beta = G_{\alpha\beta},\ p^\alpha_i\, p^i_\beta = \delta^\alpha_\beta,$$

$$p^\alpha_i \overset{def}{=} G^{\alpha\beta}\, G_{ij}\, p^j_\beta\,. \qquad ...(1.4)$$

The connections Γ^i of $K^{(2)}_n$ and Γ^α of $K^{(2)}_m$ are given as

$$2\Gamma^i = (2A_{kj}x'^j - B_{(k)})\, G^{ki},\ 2\Gamma^\alpha = (2a_{\beta\gamma}u'^\gamma - b_{(\beta)})\, G^{\beta\alpha} \qquad ...(1.5)$$

where

$$A_{kj} = \partial_j A_k$$

and

$$a_{\beta\gamma} = \partial_\gamma a_\beta.$$

The covariant differential of a contravariant vector field $v^i(x, x')$ homogeneous of degree zero with respect to x'^i is defined as (Kawaguchi 1938)

$$\delta v^i = dv^i + \Gamma^i_{jk}\, v^j dx^k \qquad ...(1.6)$$

where

$$\Gamma^i_{jk} = \partial^2\Gamma^i/\partial x'^j\,\partial x'^k = \Gamma^i_{(j)(k)} = \Gamma^i_{kj}\,. \qquad ...(1.7)$$

If v^α be a vector field in $K^{(2)}_m$ such that $v^i = p^i_\alpha\, v^\alpha$ then the induced covariant differential $\tilde{\delta}\, v^\alpha\, (= p^\alpha_i\ \delta v^i)$ is given by

$$\tilde{\delta}\, v^\alpha = dv^\alpha + \tilde{\Gamma}^\alpha_{\beta\gamma}\, v^\beta du^\gamma \qquad ...(1.8)$$

in which

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = p^\alpha_i\, (p^i_{\beta\gamma} + \Gamma^i_{jk}\, p^j_\beta\, p^k_\gamma),\ p^i_{\beta\gamma} = \partial_\gamma\, p^i_\beta\,. \qquad ...(1.9)$$

Yoshida (1967) has defined

$$\overset{\circ}{H}{}^i_{\alpha\beta} = \overset{\circ}{D}_\beta\, p^i_\alpha \overset{def}{=} p^i_{\alpha\beta} + \Gamma^i_{jk} p^j_\beta\, p^k_\alpha - \tilde{\Gamma}^\gamma_{\alpha\beta}\, p^i_\gamma\,. \qquad ...(1.10)$$

2. Lie Derivatives in the Space $K^{(2)}_n$

Les us consider an infinitesimal transformation of the type

$$\bar{x}^i = x^i + v^i(x)\, d\tau \qquad ...(2.1)$$

where x^i and $\bar{x}^i$ are coordinates of $K_n^{(2)}$ and $\bar{K}_n^{(2)}$ respectively, $v^i(x)$ are the contravariant components of the vectors along which the deformation is considered and $d\tau$ is an infinitesimal parameter. The corresponding variations in x'^i and x''^i are given by

$$\text{(a)}\quad \bar{x}'^i = x'^i + \partial_j v^i x'^j d\tau,$$

$$\text{(b)}\quad \bar{x}''^i = x''^i + [\partial_j v^i x''^j + \partial_k \partial_j v^i x'^k x'^j]\, d\tau. \qquad ...(2.2)$$

Differentiating (2.1) with respect to x^j, we get

$$\partial_j \bar{x}^i = \delta^i_j + \partial_j v^i d\tau. \qquad ...(2.3)$$

Rewriting (2.1) as $x^i = \bar{x}^i - v^i(x)\, d\tau$ and differentiating (2.1) with respect to $\bar{x}^j$ and using $\partial_j v^i d\tau = \bar{\partial}_j v^i\, d\tau$, $\bar{\partial}_j = \partial/\partial\bar{x}^j$, we have

$$\bar{\partial}_j x^i = \delta^i_j - \partial_j v^i\, d\tau. \qquad ...(2.4)$$

Let $X^i(x, x', x'')$ be a vector field defined over $K_n^{(2)}$, the value of the function $X^i(x, x', x'')$ at $\bar{x}$ (using point transformation) can be written (by Taylor's expansion theorem) as

$$\begin{aligned} X^i(\bar{x}, \bar{x}', \bar{x}'') = X^i(x, x', x'') + [&v^j \partial_j X^i \\ &+ X^i_{(k)} \partial_j v^k x'^j + X^i_{((k))} (x''^j \partial_j v^k \\ &+ \partial_l \partial_j v^k x'^l x'^j)]\, d\tau \end{aligned} \qquad ...(2.5)$$

in which only the first order of $d\tau$ has been considered. Using

$$\overset{v}{d} X^i \overset{def}{=} X^i(\bar{x}, \bar{x}', \bar{x}'') - X^i(x, x', x''), \qquad ...(2.6)$$

we have

$$\begin{aligned} \overset{v}{d} X^i = [&v^j \partial_j X^i + X^i_{(k)} \partial_j v^k x'^j + X^i_{((k))} \\ &(x''^j \partial_j v^k + \partial_l \partial_j v^k x'^l x'^j)]\, d\tau. \end{aligned} \qquad ...(2.7)$$

Considering the equation (2.1) as the coordinate transformation and $\bar{X}^i(\bar{x}, \bar{x}', \bar{x}'')$ as the component of the vector field $X^i(x, x', x'')$ in the $\bar{x}$ coordinate system, we get after using (2.3),

$$\bar{X}^i(\bar{x}, \bar{x}', \bar{x}'') = \partial_j \bar{x}^i X^j = X^i(x, x', x'') + \partial_j v^i X^j d\tau. \qquad ...(2.8)$$

Also, we have

$$\begin{aligned} \overset{m}{d} X^i &\overset{def}{=} \bar{X}^i(\bar{x}, \bar{x}', \bar{x}'') - X^i(x, x', x'') \\ &= \partial_j v^i X^j d\tau. \end{aligned} \qquad ...(2.9)$$

Now the Lie derivative of the vector field $X^i(x, x', x'')$ in $K_n^{(2)}$ is defined as (Rund 1959, Yano 1957)

$$\underset{v}{\mathcal{L}} X^i \overset{def}{=} \frac{d^v X^i - d^m X^i}{d\tau} \qquad \text{...(2.10)}$$

which gives

$$\underset{v}{\mathcal{L}} X^i(x, x', x'') = v^j \partial_j X^i + X^i_{(k)} \partial_j v^k x'^j$$
$$+ X^i_{((k))} (x''^j \partial_j v^k + \partial_l \partial_j v^k x'^l x'^j) - \partial_j v^i X^j. \qquad \text{...(2.11)}$$

In a similar way, the Lie derivatives of a covariant vector $X_i(x, x', x'')$ and a scalar $N(x, x', x'')$ are given by

$$\underset{v}{\mathcal{L}} X_i(x, x', x'') = v^j \partial_j X_i + X_{i(k)} \partial_j v^k x'^j$$
$$+ X_{i((k))}(x''^j \partial_j v^k + \partial_l \partial_j v^k x'^l x'^j) + \partial_i v^j X_j \qquad \text{...(2.12)}$$

and

$$\underset{v}{\mathcal{L}} N(x, x', x'') = v^j \partial_j N + N_{(k)} \partial_j v^k x'^j$$
$$+ N_{((k))}(x''^j \partial_j v^k + \partial_l \partial_j v^k x'^l x'^j). \qquad \text{...(2.13)}$$

Also, we have

$$\underset{v}{\mathcal{L}} x'^i = 0, \quad \underset{v}{\mathcal{L}} x''^i = 0. \qquad \text{...(2.14)}$$

Since the arc length s given by (1.1) is scalar, $A_i(x, x')$ is a vector. The Lie-derivative of A_i (in view of (2.12)) is given by

$$\underset{v}{\mathcal{L}} A_i(x, x') = v^j \partial_j A_i + A_{i(k)} \partial_j v^k x'^j + \partial_i v^j A_j. \qquad \text{...(2.15)}$$

In consequence of (2.13), the Lie derivative of the scalar $B(x, x')$ can be written as

$$\underset{v}{\mathcal{L}} B(x, x') = v^j \partial_j B + B_{(k)} \partial_j v^k x'^j. \qquad \text{...(2.16)}$$

Further, the Lie-derivatives of $A_{i(l)}$ and fundamental tensor $G_{ij}(x, x')$ are given by

$$\underset{v}{\mathcal{L}} A_{i(l)} = \partial_j A_{i(l)} v^j + A_{i(l)(k)} \partial_j v^k x'^j$$
$$+ \partial_i v^j A_{j(l)} + \partial_l v^j A_{i(j)} \qquad \text{...(2.17)}$$

and

$$\underset{v}{\mathcal{L}} G_{il} = \partial_j G_{il} v^j + G_{il(k)} \partial_j v^k x'^j$$
$$+ \partial_i v^j G_{jl} + \partial_l v^j G_{ij} \qquad \text{...(2.18)}$$

where in deducing (2.18), we have used eqns. (1.3) and (2.17).

We shall now obtain the Lie-derivative of the connection parameter

$\Gamma^i_{jk}(x, x')$. Let $\Gamma^i_{jk}(\bar{x}, \bar{x}')$ and $\bar{\Gamma}^i_{jk}(\bar{x}, \bar{x}')$

be the point and coordinate transformations of the connection parameter

$\Gamma^i_{jk}(x, x')$ of $K^{(2)}_n$.

Now we have

$$\overset{v}{d}\,\Gamma^i_{jk} = [v^h \partial_h \Gamma^i_{jk} + \Gamma^i_{jk(r)} \partial_h v^r x'^h]\, d\tau. \quad \ldots(2.19)$$

Since Γ^i_{jk} are the functions of x, x' only, the law of transformation of this connection parameter will be the same as in Finsler space (Rund 1959). We therefore write

$$\Gamma^i_{jk} + \bar{\partial}_r x^i (\partial_j \partial_k \bar{x}^r + \bar{\Gamma}^r_{st} \partial_j \bar{x}^s \partial_k \bar{x}^t). \quad \ldots(2.20)$$

On using eqns. (2.3) and (2.20), we get

$$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk} + (\partial_i \partial_k v^i - \partial_r v^i \Gamma^r_{jk} + \partial_k v^r \Gamma^i_{jr} + \partial_j v^r \Gamma^i_{rk})\, d\tau.$$

Thus,

$$\overset{m}{d}\,\Gamma^i_{jk} = \bar{\Gamma}^i_{jk} - \Gamma^i_{jk}$$

$$= -(\partial_j \partial_k v^i - \partial_r v^i \Gamma^r_{jk} + \partial_k v^r \Gamma^i_{jr} + \partial_j v^r \Gamma^i_{rk})\, d\tau. \quad \ldots(2.21)$$

From eqns. (2.10), (2.19) and (2.21), we get

$$\underset{v}{\mathcal{L}}\,\Gamma^i_{jk} = v^h \partial_h \Gamma^i_{jk} + \Gamma^i_{jk(r)} \partial_h v^r x'^h$$

$$+ \partial_j \partial_k v^i - \partial_r v^i \Gamma^r_{jk} + \partial_k v^r \Gamma^i_{jr} + \partial_j v^r \Gamma^i_{rk}. \quad \ldots(2.22)$$

3. Lie Derivatives in the Subspace $K^{(2)}_m$

We consider an infinitesimal transformation of the same type as (2.1), which carries the point x of the subspace $K^{(2)}_m : x^i = x^i(u^\alpha)$ to the neighbouring point $\bar{x}$ of a subspace $\bar{K}^{(2)}_{(m)} : \bar{x}^i = \bar{x}^i(u^\alpha)$ where u^α are fixed under the transformation. The projection factor p^i_α is transformed as

$$\bar{p}^i_\alpha = p^i_\alpha + \partial_j v^i p^j_\alpha\, d\tau = \partial_\alpha \bar{x}^i. \quad \ldots(3.1)$$

Thus, the variations of x^i and p^i_α under the transformation (2.1) are represented in the form

$$\left.\begin{aligned} \delta x^i &= \bar{x}^i - x^i = v^i d\tau \\ \delta p^i_\alpha &= \bar{p}^i_\alpha - p^i_\alpha = \partial_j v^i p^j_\alpha \, d\tau. \end{aligned}\right\} \quad \ldots(3.2)$$

Since the point and coordinate transformations of p^i_α yield the same values of $\overset{v}{d} p^i_\alpha$ and $\overset{m}{d} p^i_\alpha$ respectively, we have

$$\underset{v}{\mathcal{L}} p^i_\alpha = 0 \quad \ldots(3.3)$$

whence

$$\underset{v}{\mathcal{L}} p^i_{\alpha\beta} = 0. \quad \ldots(3.4)$$

Using (1.4) and (3.3), we obtain

$$\left.\begin{aligned} \underset{v}{\mathcal{L}} G_{\alpha\beta} &= (\underset{v}{\mathcal{L}} G_{ij})\, p^i_\alpha \, p^j_\beta \\ \underset{v}{\mathcal{L}} p^\alpha_i &= G^{\alpha\epsilon} p^j_\epsilon \, \underset{v}{\mathcal{L}} G_{ij}. \end{aligned}\right\} \quad \ldots(3.5)$$

The Lie derivative of induced connection coefficient $\tilde{\Gamma}^\alpha_{\beta\gamma}$ and the tensor $\overset{\circ}{H}{}^\alpha_{\beta\gamma}$ can be written with the help of eqns. (1.9), (1.10) and (3.4) in the following form

$$\begin{aligned} \underset{v}{\mathcal{L}} \tilde{\Gamma}^\alpha_{\beta\gamma} &= G^{\alpha\epsilon} p^j_\epsilon (\underset{v}{\mathcal{L}} G_{ij})(p^i_{\beta\gamma} + \Gamma^i_{jk} p^j_\beta \, p^k_\gamma) \\ &\quad + p^\alpha_i \, (\underset{v}{\mathcal{L}} \Gamma^i_{jk})\, p^j_\beta \, p^k_\gamma \end{aligned} \quad \ldots(3.6)$$

and

$$\underset{v}{\mathcal{L}} \overset{\circ}{H}{}^i_{\beta\gamma} = (\underset{v}{\mathcal{L}} \Gamma^i_{jk})\, p^j_\beta \, p^k_\gamma - p^i_\alpha \underset{v}{\mathcal{L}} \tilde{\Gamma}^\alpha_{\beta\gamma}. \quad \ldots(3.7)$$

4. Motions in the Space and Subspace

We define (Yano 1957, Rund 1959) the following :

Definition 4.1 — An infinitesimal point transformation is called a motion if the distance between two points remains unchanged under the transformation.

Thus, we have (Watanabe 1962) the following:

Theorem 4.1 — The necessary and sufficient condition that the transformation (2.1) be a motion is that

$$\underset{v}{\mathcal{L}} F(x, x', x'') = \underset{v}{\mathcal{L}}(A_i x''^i + B) = 0. \qquad ...(4.1)$$

Thus eqn. (4.1) implies that

$$\underset{v}{\mathcal{L}} A_{i(j)} = 0, \quad \underset{v}{\mathcal{L}} B = 0 \qquad ...(4.2)$$

which gives

$$\underset{v}{\mathcal{L}} G_{ij} = 0, \quad \underset{v}{\mathcal{L}} B = 0. \qquad ...(4.3)$$

Equations (4.3) are called as Killing equations and the vector v^i is called a Killing vector for the transformation (2.1) in special Kawaguchi space.

Further, we have the following :

Theorem 4.2 — If a transformation is a motion in the space $K_n^{(2)}$, then it is a motion in the subspace $K_m^{(2)}$.

PROOF : Let the transformation (2.1) be a motion in $K_n^{(2)}$, therefore

$$\underset{v}{\mathcal{L}} (A_i x''^i + B) = 0.$$

Since,

$$x'^i = p^i_\alpha u'^\alpha \text{ and } x''^i = p^i_\sigma u''^\alpha + p^i_{\alpha\beta} u'^\alpha u'^\beta,$$

therefore,

$$A_i x''^i + B = a_\alpha u''^\alpha + b$$

where

$$a_\alpha = A_i p^i_\alpha, \; b = A_i p^i_{\alpha\beta} u'^\alpha u'^\beta + B.$$

Thus,

$$\underset{v}{\mathcal{L}} (a_\alpha u''^\alpha + b) = 0 \qquad ...(4.4)$$

which is a required condition for a motion in the subspace $K_m^{(2)}$.

From eqn. (4.3) and Theorem 4.2, we have the following :

Theorem 4.3 — The Killing equations hold in $K_n^{(2)}$, implies that, these also hold in the subspace $K_m^{(2)}$, that is

$$\underset{v}{\mathcal{L}}\, G_{\alpha\beta} = 0, \quad \underset{v}{\mathcal{L}}\, b = 0. \qquad \ldots(4.5)$$

For an affine motion, we define (Yano 1957) the following :

Definition 4.2 — The infinitesimal point transformation (2.1) is called an affine motion in $K^{(2)}_n$ if it transforms a parallel vector field into a parallel vector field.

For a parallel vector field $V^i(x, x')$ of $K^{(2)}_n$, we have

$$\delta v^i(x, x') = dV^i(x, x') + \Gamma^i_{jk}(x, x')\, V^j dx^k = 0. \qquad \ldots(4.6)$$

The connection parameters in the point and coordinate transformations are equal in the case of affine motion in $K^{(2)}_n$ (Yano 1957), that is

$$\Gamma^i_{jk}(\bar{x}, \bar{x}') = \bar{\Gamma}^i_{jk}(\bar{x}, \bar{x}'). \qquad \ldots(4.7)$$

Hence, we have (Watanabe and Yoshida 1963) the following :

Theorem 4.4 — If the transformation (2.1) be an affine motion in $K^{(2)}_n$, then it is necessary and sufficient that

$$\underset{v}{\mathcal{L}}\, \Gamma^i_{jk} = 0. \qquad \ldots(4.8)$$

Theorem 4.5 — If a transformation is an affine motion in $K^{(2)}_n$, then the necessary and sufficient conditions that it is an affine motion in $K^{(2)}_m$ are

$$\underset{v}{\mathcal{L}}\, \overset{\circ}{H}{}^i_{\beta\gamma} = 0. \qquad \ldots(4.9)$$

PROOF : If the transformation (2.1) be an affine motion in $K^{(2)}_n$, then using eqns. (1.10), (3.4) and (4.8), we have

$$\underset{v}{\mathcal{L}}\, \overset{\circ}{H}{}^i_{\beta\gamma} + p^i_\alpha\, \underset{v}{\mathcal{L}}\, \tilde{\Gamma}^\alpha_{\beta\gamma} = 0. \qquad \ldots(4.10)$$

For an affine motion in $K^{(2)}_m$, using $\underset{v}{\mathcal{L}}\, \overset{\circ}{H}{}^i_{\beta\gamma} = 0$, we have

$$\underset{v}{\mathcal{L}}\, \tilde{\Gamma}^\alpha_{\beta\gamma} = 0. \qquad \ldots(4.11)$$

Conversely, if the transformation is an affine motion in $K_m^{(2)}$, then using eqns. (1.10), (4.9) and (4.11) we obtain $\underset{v}{\mathcal{L}} \Gamma^i_{jk} = 0$, which proves the theorem.

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