A HILLE-YOSIDA-PHILLIPS TYPE OF THEOREM FOR SEMI-GROUPS IN A LOCALLY CONVEX SPACE

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A theorem on the necessary and sufficient conditions for the equi-continuity of a discrete semi-group is given in a locally convex space.

Let X be a locally convex Hausdorff linear topological space and $\{T_t : t \geqslant 0\}$ a one parameter family of continuous linear operators in $L(X, X)$ such that

$$T_t T_s = T_{t+s}, \quad T_0 = I, \quad \lim_{t \to t_0} T_t x = T_{t_0} x, \quad \text{for } t_0 > 0$$

and $\qquad x \in X.$

Such a family T_t is called an equi-continuous semi-group of class C_0 [*see* Yosida 1971] if for any continuous semi-norm p on X there exists a continuous semi-norm q on X such that

$$p(T_t x) \leqslant q(x), \quad \text{for all } t \geqslant 0 \text{ and all } x \in X.$$

We consider a discrete semi-group $T_k = B^k$ $(k = 1, 2, 3, \ldots)$ on a sequentially complete, locally convex Hausdorff linear topological space X, where $B \in L(X, X)$.

In this paper we extend the theorems for discrete semi-groups proved by Gibson (1972) for equi-continuous discrete semi-groups in X In the following $R_z = (I - zB)^{-1}$, where z is a complex number will stand for the resolvent of B.

Theorem 1 — If B is a continuous linear operator on X then a necessary and sufficient condition that the semi-group $T_k = B^k$ be equi-continuous is that

$$\left\{\frac{(1 - |z|)^{n+1}}{|z|^n} (R_z - I)^n R_z ; n = 0, 1, 2, \ldots \text{ and } 0 < |z| < 1\right\} \quad \ldots(1)$$

be a equi-continuous family.

PROOF : We first prove that the condition (1) is necessary. Since $T_k = B^k$ is equi-continuous given a semi-norm (continuous) p on X there exists a continuous semi-norm q on X such that

$$p(B^k x) \leqslant q(x) \quad \text{for } k = 1, 2, 3, \ldots$$

the resolvent R_z may be written as

$$R_z x = (1 - zB)^{-1} x = \sum_{k=0}^{\infty} z^k B^k x \text{ for } x \in X \text{ and } |z| < 1$$

and hence differentiating term by term, we obtain

$$R_z^{(n)} x = \sum_{k=n}^{\infty} k(k-1) \dots (k-n+1) z^{k-n} B^k x.$$

Here the term by term differentiation is justified because of the fact that $B^k x$ is equi-bounded in k.

Therefore,

$$p(R_z^{(n)} x) \leqslant \sum_{k=n}^{\infty} k(k-1) \dots (k-n+1) |z|^{k-n} p(B^k x)$$

$$\leqslant \sum_{k=n}^{\infty} k(k-1) \dots (k-n+1) |z|^{k-n} q(x)$$

$$= \frac{n!}{(1-|z|)^{n+1}} q(x)$$

i.e. $$p\left(\frac{(1-|z|)^{n+1}}{n!} R_z^{(n)} x\right) \leqslant q(x) \qquad \dots(2)$$

which implies that the family as mentioned in the statement of Theorem 1, is a equi-continuous family, because of the following fact.

By lemma 2, in Gibson (1972) which can be easily extended to the case when X is a locally convex space,

$$R_z^{(n)} x = \frac{n!(R_z - I)^n}{z^n} R_z x \qquad \dots(3)$$

for $x \in X$ and $|z| < 1$ and $n = 0, 1, 2, \dots$ which completes the proof of the necessity part.

To prove the sufficiency of (1) define $S_z(k)$ for $|z| < 1$ and $k = 0, 1, 2, \dots$ by

$$S_z(k) x = z^{-k} (R_z - I)^k R_z x \quad \text{for } x \in X.$$

Then (1) implies that, given a continuous semi-norm p on X there exists a continuous semi-norm q on X such that

$$p(S_z(k)x) \leqslant \frac{1}{(1-|z|)^{k+1}} q(x)$$

for $|z| < 1$ and $k = 0, 1, 2, \dots$.

Since B and R_z commute, we have

$$\begin{aligned} S_z(k)\,x &= z^{-k}\,[\{I-(I-zB)\}\,R_z]^k\,R_z x, \quad x \in X \\ &= z^{-k}(zBR_z)^k\,R_z x \\ &= B^k R_z^{k+1}\,x. \end{aligned}$$

From which it follows that for $x \in X$

$$\begin{aligned} p\,[(B^k - S_z(k))x] &= p\,[S_z(k)\,\{(I-zB)^{k+1} - I\}\,x] \\ &\leqq \frac{1}{(1-|z|)^{k+1}}\,q([(I-zB)^{k+1}-I]\,x) \\ &= \frac{1}{(1-|z|)^{k+1}}\,q\Big[\sum_{j=1}^{k+1}\binom{k+1}{j}(-zB)^j x\Big]. \qquad \ldots(4) \end{aligned}$$

Now since B^j are continuous linear operators on X for each j there exists c_j's such that $q(B^j x) \leqslant c_j q(x)$ from which it follows from (4) that

$$p\,[(B^k - S_z(k))\,x] \leqslant \frac{|z|}{(1-|z|)^{k+1}}\sum_{j=1}^{k+1}\left|\binom{k+1}{j}c_j\right| q(x).$$

Putting L_k as the constant $L_k = \sum_{j=1}^{k+1}\left|\binom{k+1}{j}c_j\right|$

we get

$$p[(B^k - S_z(k))\,x] \leqslant \frac{L_k\,|z|}{(1-|z|)^{k+1}}\,q(x)$$

and hence

$$\begin{aligned} p(B^k x) &= p\,[(B^k - S_z(k))\,x + S_z(k)\,x] \\ &\leqslant p\,[(B^k - S_z(k))\,x] + p\,[S_z(k)x] \\ &\leqslant \frac{L_k\,|z|+1}{(1-|z|)^{k+1}}\,q(x). \end{aligned}$$

In particular, for $z = 0$ and $k = 0, 1, 2, \ldots$ we obtain

$$p(B^k x) \leqslant q(x)$$

which proves the sufficiency part of the theorem. If Y is a Banach space, we say that $T_k = B^k$ is a quasibounded discrete semi-group on Y if $\| B^k \|_Y \leqslant M\omega^k$ for real

$\omega > 0$ and $k = 0, 1, 2, \ldots$ with the analogy of the quasibounded discrete semi-group we give the following theorem, when X is a locally convex linear topological space which is Hausdorff and sequentially complete.

Theorem 2 — If B is a continuous linear operator on X, then a necessary and sufficient condition that $\omega^{-k}B^k$ for $\omega > 0$ be equi-continuous is that

$$\left\{\frac{(1 - \omega \mid z \mid)^{n+1}}{\omega^n \mid z \mid^n} (R_z - I)^n R_z; \mid z \mid < \frac{1}{\omega} \text{ and } n = 0, 1, 2, \ldots\right\}$$

be a equi-continuous family.

PROOF: We only need to show that $B_1 = \frac{1}{\omega} B$ and $z_1 = \omega z$ satisfy the condition of Theorem 1, which is very clear.

REFERENCES

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