ON GENERALIZED ADJOINTS

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(*Received* 27 *June* 1977; *after revision* 14 *April* 1978)

In this paper, the notion of generalized adjoints of a linear operator on a sub-space of a Hilbert space into the Hilbert space as single valued operators, is introduced and the properties of these generalized adjoints, when the domain of the operator is not dense, are studied.

John von Neumann (1950) introduced the notion of generalized adjoints as multivalued operators. In this paper we introduce generalized adjoints as single-valued operators. The properties of such operators, when the domain is dense, can be found in Riesz and Nagy (1956). The purpose of this paper is to investigate the properties of adjoints when the domain is not dense. We also investigate certain cardinality properties.

In what follows we adopt the following :

Notation — X is a Hilbert space and (x, y) is the inner product of x and y ;

$H = X \times X$ with the usual Hilbert space structure;

elements of H are denoted by $\langle x, y \rangle$, $x, y \in X$;

$V : H \rightarrow H$ is defined as $V \langle x, y \rangle = \langle y, -x \rangle$;

M is a subspace of X, $\bar{M}$ is the closure of M and P is the orthogonal projection of X onto $\bar{M}$;

$T : M \rightarrow X$ is linear, $\mathcal{R}(T)$ is the range of T, $G(T)$ is the graph of T and $\bar{G}(T)$ is the closure of $G(T)$ in H;

$$N = \{y \mid \exists\, y^* \in X \ni (Tx, y) = (x, y^*) \;\forall\, x \in M\}.$$

Definition 1 — $T^* : N \rightarrow \bar{M}$ is defined as $T^*y = Py^*$ where

$$(Tx, y) = (x, y^*) \;\forall\, x \in M.$$

T^* is well defined for if $u, v \in X$ are such that $(Tx, y) = (x, u)$ and $(Tx, y) = (x, v) \;\forall\, x$ in M then $(x, u) = (x, v) \;\forall\, x$ in M so that $u - v \in M^{\perp}$ and hence $Pu = Pv$. It can

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be easily seen that T^* is linear and closed. Now, from Theorems 4.2–D and 4.2–I of Taylor (1958), it follows that

(I) T^* is continuous iff N is closed.

Also we observe that

$$\overline{\mathcal{R}(T)} \subset N \Leftrightarrow N = X \text{ since } (\mathcal{R}(T))^{\perp} \subset N.$$

Notation — $\hat{G}(T^*) = \{\langle y, y^* \rangle \mid (Tx, y) = (x, y^*) \ \forall \ x \in M\}$. It may be noted that $\hat{G}(T^*)$ is closed.

Theorem 2 — If T is continuous then $N = X$, T^* is continuous and $\| T^* \| = \| T \|$.

PROOF : First assume that M is closed. Let $A = TP$ so that A is continuous on X and $\| A \| = \| T \|$.

Since $(Tx, y) = (TPx, y) = (Ax, y) = (x, A^*y) \ \forall \ x \in M$ and $y \in X$, we have $X = N$. Now

$$y \in X \Rightarrow (x, A^*y) = (Ax, y) = 0 \text{ if } x \in M^{\perp}$$
$$\Rightarrow A^*y \in M \Rightarrow A^*y = T^*y.$$

Thus $T^* = A^*$ is continuous and $\| T^* \| = \| A^* \| = \| A \| = \| T \|$. Now we assume that M is not closed. Let $\tilde{T}$ denote the continuous extension of T to $\bar{M}$. It can be seen that $(\tilde{T})^* = T^*$ so that from the above discussion the result follows.

Theorem 3 — If $N = X$, then T and T^* are continuous and hence $\| T \| = \| T^* \|$.

PROOF : Since T^* is closed and $N = X$, it follows that T^* is continuous. Now T is continuous since it is the restriction of T^{**} (which is continuous) to M.

Note : From the above two theorems it follows that T is continuous iff $N = X$.

Definition 4 — A linear operator $A : N \rightarrow X$ is called an adjoint of T if

$$(Tx, y) = (x, Ay) \ \forall \ x \in M, y \in N.$$

Note : T^* is an adjoint of T. We call T^* the principal adjoint of T. We observe that A is an adjoint of T iff A is of the form $T^* + B$ where $B : N \rightarrow M^{\perp}$ is linear.

Theorem 5 — If T has a continuous adjoint, then T^* is continuous.

PROOF : Follows from the above note, by noting that $\| T^*x \| \leqslant \| Ax \|$ when $x \in N$.

We note the following :

(i) T has a closed linear extension iff N is dense in X. (*see* John von Neumann 1950, Theorem 13.14, Corollary 2).

(ii) If T has a continuous adjoint and $\mathcal{R}(T) \subset \bar{N}$ then T is continuous. [Since $(\mathcal{R}(T))^{\perp} \subset N$ it follows that $\bar{N} = X$. From Theorem 5, T^* is continuous. Hence, from (*I*), $N = X$ so that from Theorem 3, T is continuous].

(iii) If T is unbounded and N is dense in X then T cannot have a continuous adjoint. [This follows from (ii) since $\mathcal{R}(T) \subset X = \bar{N}$].

(iv) If T has a closed linear extension and T is unbounded then T can not have a continuous adjoint. [This is immediate from (i) and (iii)].

Theorem 6 — Every adjoint (of T) which has a closed linear extension is closed.

PROOF : Let A be an adjoint of T with the minimal closed linear extension $\bar{A}$. Then $G(\bar{A}) = \bar{G}(A)$. From the very definition of $\hat{G}(T^*)$, it follows that $G(A) \subset \hat{G}(T^*)$ which implies that $\bar{G}(A) \subset \hat{G}(T^*)$ since the latter is closed. Hence $G(\bar{A}) \subset \hat{G}(T^*)$, consequently, domain of $\bar{A}$ is a subset of N = domain of A and hence $\bar{A} = A$.

Certain results on adjoints, with $\bar{M} = X$ assumed, were proved in Riesz and Nagy (1956, §117). In particular, in that case T^{**} is the closed linear extension of T. The following example shows that this is not true in general.

Example — Let $X = l^2$,

$M = \{x \in l^2 \mid x = \{x_k\},\ x_1 = 0 \text{ and } x_k \neq 0 \text{ for only finitely many } k\}$,

$T : M \to X$ be the backward shift operator, more specifically,

$T(\{x_k\}) = \{y_k\}$ where $y_k = x_{k+1}$ for $k = 1, 2, \ldots$.

Then $N = X$ and $T^* : N \to X$ is the forward shift operator, that is

$T^*(\{y_k\}) = \{z_k\}$ where $z_1 = 0$ and $z_{k+1} = y_k$ for $k = 1, 2, \ldots$

so that $T^{**} : X \to X$ is the backward shift operator. The continuous extension of T to $\bar{M}$ is clearly the minimal closed linear extension $\bar{T}$ of T. Thus T^{**} is a closed linear extension of T different from $\bar{T}$.

[It may be noted that the operator $A : X \to X$ given by $A(\{y_k\}) = \{z_k\}$ where $z_1 = y_1$ and $z_{k+1} = y_k$ for $k = 1, 2, \ldots$ is also an adjoint of T, $A^* : X \to X$ is given by $A^*(\{x_k\}) = \{y_k\}$ where $y_1 = x_1 + x_2$ and $y_k = x_{k+1}$ for $k = 2, 3, \ldots$ and that A^* is a closed linear extension of T different from $\bar{T}$].

This example gives rise to a strong suspicion that there are striking contrasts if we assume that M is not dense in X. That it is the case is evident from the following three theorems.

Theorem 7 — If T has a closed linear extension and M is not dense in X then the adjoint A^* of A is a proper extension of $\bar{T}$ (the minimal closed linear extension of T) where A is any closed adjoint of T.

PROOF: From Riesz and Nagy (1956, §117), we have

$$H = G(A^*) + V(G(A)) \text{ and } H = \bar{G}(T) + V(\hat{G}(T^*)).$$

Since $\bar{M} \neq X$, $G(A)$ is a proper subspace of $\hat{G}(T^*)$.

Hence $\bar{G}(T)$ is a proper subspace of $G(A^*)$.

Theorem 8 — If $\bar{M} \neq X$ and N is infinite dimensional, then there exists an adjoint of T which has no closed linear extension.

PROOF: N is infinite dimensional $\Rightarrow G(T^*)$ is so. Let $\{\langle y_n, T^*y_n\rangle\}$ be an orthonormal sequence in $G(T^*)$. Write $z_n = \frac{1}{n} y_n$. Let S be a Hamel basis of N containing $\{z_n\}$. Let $0 \neq x_0 \in M^\perp$. Define $B(z_n) = x_0 \ \forall\ n$, $B(y) = 0$ if $y \in S$, $y \neq z_n \ \forall\ n$ and extend it linearly to N. Let $A = T^* + B$. Then, clearly, A is an adjoint of T, $\langle 0, x_0\rangle \in \bar{G}(A)$. Hence, A has no closed linear extension.

Theorem 9 — If $\bar{M} \neq X$, N is closed and $N \neq \{0\}$ then there exists a closed adjoint A of T such that $A \neq T^*$.

PROOF: Let $0 \neq y_0 \in N$, Y the subspace generated by y_0. Let Z be the orthogonal complement of Y in N so that $N = Y \oplus Z$. Let y_0^* be such that $(Tx, y_0) = (x, y_0^*) \ \forall\ x \in M$ and $y_0^* \neq T^*y_0$. Such a y_0^* exists since M is not dense in X.

Define $A : N \to X$ by $Ay = \alpha y_0^* + T^*z$ where $y = \alpha y_0 + z$, $z \in Z$.

Then A is a closed adjoint of T, different from T^*.

Now we prove certain cardinality theorems on adjoints.

Theorem 10 — Let $M^\perp$ be infinite dimensional. Then the Hamel dimension of N is greater than or equal to the Hilbert dimension of $M^\perp$ iff $\exists$ an adjoint A of $T \ni \bar{G}(A) \equiv \hat{G}(T^*)$.

PROOF: Suppose that the Hamel dimension of N is greater than or equal to the Hilbert dimension of $M^\perp$. Let S be a Hamel basis of N and E a maximal orthonormal set in $M^\perp$. We may assume that $\| \langle x, T^*x \rangle \| = 1 \ \forall\ x \in S$. Let K be a subset of S such that K and E have the same cardinality. Write $K = \bigcup_{x \in E} K_x$, where K_x is countably infinite for each x in E and $K_x \cap K_y = \phi$ if $x \neq y$. Let $K_x = \{x_1, x_2, \ldots\}$. Define B on N linearly as follows :

$$By = \begin{cases} nx & \text{if } y = x_n \text{ for some } x \in E \text{ and } n \\ 0 & \text{if } y \in S - K. \end{cases}$$

Define $A = T^* + B$. Now let $x \in E$ and $y_n = \frac{1}{n} x_n$. Then

$$\|\langle x_n, T^*x_n \rangle \| = 1 \Rightarrow \| \langle y_n, T^*y_n \rangle \| = \frac{1}{n} \Rightarrow \| y_n \|^2 + \| T^*y_n \|^2 = \frac{1}{n^2}$$

$$\Rightarrow y_n \to 0,\ T^*y_n \to 0 \text{ as } n \to \infty$$

$$\Rightarrow Ay_n = T^*y_n + By_n = T^*y_n + x \to x \text{ as } n \to \infty$$

$$\Rightarrow \langle y_n, Ay_n \rangle \to \langle 0, x \rangle \text{ as } n \to \infty$$

$$\Rightarrow \langle 0, x \rangle \in \bar{G}(A).$$

Thus $\langle 0, x \rangle \in \bar{G}(A) \ \forall\ x \in E$ and hence $\langle 0, x \rangle \in \bar{G}(A) \ \forall\ x \in M^\perp$.

Hence it follows that $\bar{G}(A) = \hat{G}(T^*)$.

Now, suppose conversely, that $\bar{G}(A) = \hat{G}(T^*)$ for some adjoint A of T.
Let $A = T^* + B$ where $B : N \to M^\perp$, F a maximal orthonormal set in $\mathcal{R}(B)$. Extend F to a maximal orthonormal set E in $M^\perp$. Now

$$\bar{G}(A) = \hat{G}(T^*) \Rightarrow \langle 0, x \rangle \in \bar{G}(A) \ \forall\ x \in M^\perp$$

$$\Rightarrow \langle 0, x \rangle \in \bar{G}(A) \ \forall\ x \in E$$

$$\Rightarrow \langle 0, x \rangle \in \bar{G}(B) \ \forall\ x \in E. \qquad \ldots(1)$$

Now suppose that Hamel dimension of N is less than the Hilbert dimension of $M^\perp$. Then

$$\text{card } F \leqslant \text{Hamel dim } \mathcal{R}(B) \leqslant \text{Hamel dim } N$$

$$< \text{Hilbert dim } M^\perp = \text{card } E.$$

Hence F is a proper subset of E. Let $x \in E - F$.

Then $\langle 0, x \rangle \notin \bar{G}(B)$ since $x \notin \overline{\mathcal{R}(B)}$.
This contradicts (1).

In proving the second part of the above theorem, infinite dimensionality of $M^\perp$ is not used. Hence, that part is true in general. When M is finite dimensional, we can say some thing more.

Theorem 11 — Let $M^{\perp}$ be finite dimensional and $\bar{M} \neq X$. Then N is infinite dimensional $\Leftrightarrow \exists$ an adjoint A of $T \ni \bar{G}(A) = \hat{G}(T^*)$.

PROOF : Let N be infinite dimensional, K a countably infinite subset of a Hamel basis S of N, and E a maximal orthonormal set in $M^{\perp}$. Then write $K = \bigcup_{x \in E} K_x$, where K_x is countably infinite, $K_x \cap K_y = \phi$ if $x \neq y$. Now proceed as in the proof of Theorem 10.

To prove the second part of the theorem, we proceed as follows :

N is finite dimensional $\Rightarrow N$ is closed and A is continuous

$$\Rightarrow \bar{G}(A) = G(A) \neq \hat{G}(T^*) \text{ since } \bar{M} \neq X.$$

Theorem 12 — Let $\bar{M} \neq X$, T have a closed linear extension. Then X is infinite dimensional $\Leftrightarrow \exists$ an adjoint A of $T \ni \bar{G}(A) = \hat{G}(T^*)$.

PROOF : Suppose X is infinite dimensional. Since T has a closed linear extension, N is dense in X so that N is infinite dimensional. Now

$$\text{Hamel dim } N \geqslant \text{Hilbert dim } N = \text{Hilbert dim } X \geqslant \text{Hilbert dim } M^{\perp}.$$

Now applying Theorems 10 and 11, we get the result. The second part of the theorem is trivial.

Theorem 13 — Let X be infinite dimensional and T have a closed linear extension. Then there exists an adjoint A of T such that A^* is the minimal closed linear extension of T.

PROOF : There exists an adjoint A of T such that $\bar{G}(A) = \hat{G}(T^*)$ (use Theorem 12 when $\bar{M} \neq X$). Now $\bar{G}(T)$ and $V(\hat{G}(T^*))$ are orthogonal complements and $V(\bar{G}(A))$ and $G(A^*)$ are orthogonal complements in H. Hence $\bar{G}(T) = G(A^*)$.

Remark : When X is finite dimensional, A^* is an extension of T to X for any adjoint A of T.

ACKNOWLEDGEMENT

The authors are grateful to Dr D. R. K. Sangameswara Rao, for his invaluable help in preparing this paper.

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