FLUCTUATING LAMINAR FLOW PAST A NATURALLY PERMEABLE BED

by OM PRAKASH, *Department of Mathematics, Government College, Ajmer* 305001

and

S. C. RAJVANSHI, *Department of Applied Sciences, Punjab Engineering College, Chandigarh* 160012

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Laminar flow of an incompressible fluid past a fixed infinite naturally permeable bed is considered taking the tangential velocity-slip at its surface into account. The free-stream and the suction velocities fluctuate in time in the same phase about their respective means. The tangential velocity-slip at the boundary has a moderating influence on the skin friction, reducing both the amplitude as well as the phase difference of its oscillating part. It also reduces the possibility of flow reversal in the boundary layer.

1. INTRODUCTION

The importance of understanding the effect of fluctuation in the free-stream and normal velocities on the drag at fixed boundaries needs no emphasis. Lighthill (1954) initiated the study of such phenomena by considering the effect of fluctuations in the magnitude of the velocity of the free-stream flowing past fixed solid circular boundaries. Stuart (1955) extended the classical 'asymptotic suction solution' to the case of flow past a fixed porous plate, in which the velocity of the free-stream parallel to the plate fluctuates in time about a constant mean, the suction velocity being constant and uniform. Messiha (1966) assumed that the suction velocity also oscillates about a mean, in the same phase as the free-stream velocity. The corresponding magneto-hydrodynamic flow problem has been considered by Soundalgekar (1969). Free convection flows past plane surfaces have also been studied, e.g., by Soundalgekar (1973).

In these studies, the usual 'no slip' condition has been supposed to apply at the surface of a saturated porous material also. But on account of the transfer of momentum in the streamwise direction across the boundary of the porous medium and the coupled flow taking place inside it, as also due to the smoothening of the surface by the fluid trapped in its pores, some amount of velocity slip at the surface of a saturated porous material seems inevitable. On empirical considerations, Beavers and Joseph (1967) suggested a boundary condition of the form

$$u - U = \left(\frac{\sqrt{k}}{\alpha}\right)\left(\frac{\partial u}{\partial n}\right) \qquad ...(1.1)$$

where u is the fluid velocity at the permeable interface, U the Darcy velocity inside the porous medium in the streamwise direction, $(\partial u/\partial n)$ the gradient of the streamwise component of velocity along the normal to the surface drawn into the fluid, k the permeability of the porous material and α an empirical dimensionless constant believed depending upon the nature of the porous material only. In further experiments, Beavers *et al.* (1970) found that the slip-velocity was sometimes as large as sixty per cent of the mean velocity in a channel. Taylor (1971) and Richardson (1971), in companion papers, have reported experimental and analytical results that go to support the condition (1.1).

In the present study, the laminar flow of an incompressible fluid past a fixed infinite plane porous bed of homogeneous permeability is considered. The effect of tangential velocity slip at the boundary on the velocity in the boundary layer and on the phase lead and amplitude of the oscillation of skin friction is investigated. The free-stream velocity and the suction velocity are both assumed oscillating in phase with equal frequencies about their respective mean values. The suction velocity is such that only the fluid particles in the immediate neighbourhood of the porous bed are sucked away, not disturbing the parallel potential flow of the free stream (cf. Schlichting 1968, p. 367). It is found that the tangential velocity-slip at the boundary has a moderating influence on the skin friction, reducing both the amplitude as well as the phase lead of its oscillating part. It also reduces the possibility of separation and reversion of flow near the boundary.

2. Equations of Motion

An incompressible fluid of density ρ and viscosity μ occupies semi-infinite region I of space, bounded by a naturally permeable bed of permeability k occupying the region II. Their common surface is in the plane $\bar{y} = 0$, the positive direction of $\bar{y}$-axis being along the normal drawn into the fluid. The $\bar{x}$-axis is drawn in this plane in the streamwise direction, the location of the origin being immaterial. The components of velocity in the direction of coordinate axes are $\bar{u}$ and $\bar{v}$ and the hydrodynamic pressure is $\bar{p}$; the subscripts 1 and 2 indicate their values in the two regions respectively. The variable of time is denoted by $\bar{t}$. The exact width of the permeable bed need not be specified, but it is understood to be large enough to support the substratum of fluid that gives rise to the phenomenon of tangential velocity-slip.

It is expected that in the flow induced by the uniform free-stream moving parallel to the bed with a velocity $\bar{U}(\bar{t})$, the velocity components are independent of $\bar{x}$. Therefore, in region I, the equations of continuity and motion take the form

$$\frac{\partial \bar{v}_1}{\partial \bar{y}} = 0 \qquad ...(2.1)$$

$$\rho\left[\frac{\partial \bar{u}_1}{\partial \bar{t}} + \bar{v}_1\left(\frac{\partial \bar{u}_1}{\partial \bar{y}}\right)\right] = -\frac{\partial \bar{p}_1}{\partial \bar{x}} + \mu\left(\frac{\partial^2 \bar{u}_1}{\partial \bar{y}^2}\right) \qquad ...(2.2)$$

and

$$\rho\left(\frac{d\bar{v}_1}{d\bar{t}}\right) = -\frac{\partial \bar{p}_1}{\partial \bar{y}}. \qquad ...(2.3)$$

In the free stream,

$$\frac{d\bar{U}}{d\bar{t}} = -\frac{\left(\frac{\partial \bar{p}_1}{\partial \bar{x}}\right)}{\rho}. \qquad ...(2.4)$$

Darcy's law applies in region II, giving

$$\bar{u}_2 = -k\,\frac{\left(\frac{\partial \bar{p}_2}{\partial \bar{x}}\right)}{\mu} \qquad ...(2.5)$$

$$\bar{v}_2 = -k\,\frac{\left(\frac{\partial \bar{p}_2}{\partial \bar{y}}\right)}{\mu} \qquad ...(2.6)$$

where, by the equation of continuity,

$$\frac{\partial^2 \bar{p}_2}{\partial \bar{x}^2} + \frac{\partial^2 \bar{p}_2}{\partial \bar{y}^2} = 0 \qquad ...(2.7)$$

$$\bar{u}_1(\bar{y}, \bar{t}) \to \bar{U}(\bar{t}) \text{ as } \bar{y} \to \infty. \qquad ...(2.8)$$

The continuity of normal velocity and pressure at the surface of the naturally permeable material suggest that

$$\bar{v}_1|_{\bar{y}=0+} = \bar{v}_2|_{\bar{y}=0-} \qquad ...(2.9)$$

and

$$\bar{p}_1|_{\bar{y}=0+} = \bar{p}_2|_{\bar{y}=0-}\,. \qquad ...(2.10)$$

The condition of tangential velocity-slip, on the pattern of (1.1), is

$$\bar{u}_1 - \lambda\bar{u}_2 = \left(\frac{\sqrt{k}}{\alpha}\right)\left(\frac{\partial \bar{u}_1}{\partial \bar{y}}\right) \text{ at } \bar{y} = 0 \qquad ...(2.11)$$

where α is a dimensionless constant depending upon the nature of the porous material only and λ is really equal to unity, but has been introduced to enable the recovery of the 'no slip' condition when desired.

The above equations and conditions yield

$$\bar{p}_1(\bar{x}, \bar{y}, \bar{t}) = -\rho\left[\bar{x}\left(\frac{d\bar{U}}{d\bar{t}}\right) + \bar{y}\left(\frac{d\bar{v}_1}{d\bar{t}}\right)\right] + \phi(\bar{t}) \qquad \text{...(2.12)}$$

$$\bar{p}_2(\bar{x}, \bar{y}, \bar{t}) = -\rho\bar{x}\left(\frac{d\bar{U}}{d\bar{t}}\right) - \mu\bar{v}_1\frac{\bar{y}}{k} + \phi(\bar{t}) \qquad \text{...(2.13)}$$

where $\phi(\bar{t})$ is an arbitrary function of time.

This shows that in order to maintain the flow, it is necessary to produce inside the porous medium not only a normal pressure-gradient proportional to the desired velocity of suction but also a streamwise pressure-gradient proportional to the instantaneous acceleration of the free-stream. Both $\bar{p}_1$ and $\bar{p}_2$ are seen varying linearly in the directions normal as well as parallel to that of the free-stream. At larger distances from the surface of the permeable bed, the pressure becomes infinite because an infinite mass of fluid is accelerated. Hence, to be realistic, the solution obtained here is expected to be valid on local basis only.

3. Solution for Fluctuating Flow

Let the free-stream and the suction velocities fluctuate in magnitude about their mean values as follows :

$$\bar{U}(\bar{t}) = U_0\,[1 + \epsilon \exp(i\omega\bar{t})] \qquad \text{...(3.1)}$$

and

$$\bar{v}_1(\bar{t}) = -v_0\,[1 + A\epsilon \exp(i\omega\bar{t})] \qquad \text{...(3 2)}$$

where ϵ and A are positive constants such that $\epsilon \ll 1$ and $A\epsilon \leqslant 1$. The real parts alone are to be considered in the expressions for physical quantities introduced above or arising in the sequel. A solution exists only in case of v_0 being positive, i.e., the fluid is sucked out from the main stream into the porous medium. The following dimensionless variables and parameters are now defined :

$$x = \bar{x}\,\frac{U_0\rho}{\mu},\quad y = \bar{y}\,\frac{v_0\rho}{\mu},\quad t = \bar{t}\,\frac{v_0^2\,\rho}{(4\mu)}$$

$$u = \frac{\bar{u}}{U_0},\quad U = \frac{\bar{U}}{U_0},\quad v = \frac{\bar{v}}{v_0}$$

$$p = \bar{p}(v_0^2 \rho), \; n = \frac{4\mu\omega}{(v_0^2 \rho)}$$

$$\beta = v_0 \rho \frac{\sqrt{k}}{(\alpha\mu)}, \; K = (\alpha\beta)^2.$$

Then from (2.2) and (2.4), u_1 is found to satisfy the equation

$$4\left(\frac{\partial^2 u_1}{\partial y^2}\right) - 4v_1\left(\frac{\partial u_1}{\partial y}\right) - \frac{\partial u_1}{\partial t} = -\frac{dU}{dt} \qquad \ldots(3.3)$$

and from (2.5), (2.6) and (2.13), it follows that

$$u_2 = K\frac{dU}{dt} \qquad \ldots(3.4)$$

and

$$v_2 = v_1. \qquad \ldots(3.5)$$

The boundary conditions (2.8) and (2.11) now take the form

$$u_1(y, t) \to U(t) \text{ as } y \to \infty \qquad \ldots(3.6)$$

and

$$u_1 - \lambda u_2 = \beta\left(\frac{\partial u_1}{\partial y}\right) \text{ at } y = 0. \qquad \ldots(3.7)$$

The dimensionless form of free-stream and suction velocities are

$$U(t) = 1 + \epsilon \exp(int) \qquad \ldots(3.8)$$

and

$$v_1(t) = -[1 + A\epsilon \exp(int)]. \qquad \ldots(3.9)$$

Their form suggests that

$$u_1(t) = f_1(y) + \epsilon f_2(y) \exp(int). \qquad \ldots(3.10)$$

By substituting (3.8) to (3.10) in eqns. (3.3) and (3.4) and the boundary conditions (3.6) and (3.7), neglecting the terms containing ϵ^2 and separating the non-periodic and the periodic terms, the equations and boundary conditions for the functions $f_1(y)$ and $f_2(y)$ may be obtained. They yield

$$f_1(y) = 1 - \frac{\exp(-y)}{(1+\beta)} \qquad \ldots(3.11)$$

$$f_2(y) = 1 - i4A\frac{\exp(-y)}{[n(1+\beta)]} - S\exp(-hy) \qquad \ldots(3.12)$$

where

$$h = \frac{[1 + \sqrt{(1 + in)}]}{2} = h_r + i h_i$$

$$S = \frac{\left[1 - i\left(\frac{4A}{n} + \lambda \frac{Kn}{4}\right)\right]}{(1 + \beta h)}$$

$$= S_r + i S_i.$$

Hence, the non-dimensional streamwise velocity component in the boundary layer is found to be

$$u_1(y, t) = 1 - \frac{\exp(-y)}{(1 + \beta)} + \epsilon\,[M_r \cos nt + M_i \sin nt] \qquad \text{...(3.13)}$$

where

$$M_r = 1 - (S_r \cos h_i y + S_i \sin h_i y) \exp(-h_r y)$$

$$M_i = 4A \frac{\exp(-y)}{[n(1 + \beta)]} + (S_i \cos h_i y - S_r \sin h_i y) \exp(-h_r y).$$

The skin friction at the surface of the porous material is given by

$$\bar{\tau}_0 = \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_{\bar{y}=0}$$

or, in the non-dimensional form, by

$$\tau_0 = \frac{\bar{\tau}_0}{(\rho v_0^2)} = \left(\frac{U_0}{v_0}\right)\left(\frac{\partial u}{\partial y}\right)_{y=0}.$$

Then

$$\tau_0^* = \tau_0 \frac{v_0}{U_0} = \frac{1}{(1 + \beta)} + \epsilon\,|\,B\,|\cos(nt + \theta) \qquad \text{...(3.14)}$$

where

$$|\,B\,| = \sqrt{(B_r^2 + B_i^2)}$$

$$\theta = \arctan\left(\frac{B_i}{B_r}\right)$$

$$B_r = S_r h_r - S_i h_i$$

$$B_i = \frac{4A}{[n(1 + \beta)]} + S_r h_i + S_i h_r.$$

From (3.14), the skin friction is always positive if

$$\epsilon \mid B \mid \leqslant \frac{1}{(1+\beta)}.$$

For large n, this condition takes the form

$$\epsilon \left| 1 - in \frac{\lambda K}{4} \right| \leqslant \frac{\beta}{(1+\beta)}. \qquad \text{...(3.15)}$$

When the 'no slip' boundary condition ($\beta = 0 = \lambda$) is adopted, this condition is reduced to $\epsilon \leqslant 0$, which cannot be satisfied except when the free stream and the suction velocities are steady. Hence, the skin friction is bound to become negative at some large magnitude of the frequency n or the other, when a back-flow will occur near the surface. But when the tangential velocity-slip condition (2.11) is adopted, $\lambda = 1$ while K is very small. Therefore, the condition (3.15) can be satisfied for any value of n if β is given an appropriate value by adjusting the suction velocity. Thus, the reverse flow near the boundary can be avoided whatever be the frequency of oscillation of the free-stream velocity. This may be significant in boundary layer control in fluctuating flow.

The results obtained in this section reduce to those of Messiha (1966) when $\beta = \lambda = 0$.

4. Discussion

The numerical values of h_r and h_i are as tabulated by Stuart (1955), 4λ in his work corresponding to n here. The effect of tangential velocity-slip is brought out by comparison of the skin friction and the velocity in the boundary layer obtained by taking $\lambda = 1$, $\beta = 10^{-1}$ and $K = 10^{-4}$ with those obtained on the assumption of 'no slip', i.e. $\beta = 0 = K$, $\lambda = 0$. For the material used by Beavers *et al.* (1970) in their experiments, $k = 5.1 \times 10^{-7}$ cm^2 and $\alpha = 0.1$. If the fluid be water at 20°C, of viscosity 0.01 gm/cm sec and the mean suction rate be 0.14 cm per second, then the slip-parameters take the values as assumed here. Figure 1 shows $\mid B \mid$ plotted againt n. The amplitude through which the skin friction fluctuates is seen increasing with A or the amplitude through which the suction velocity also fluctuates in phase with the free stream velocity. For any particular value of A, $\mid B \mid$ is seen reduced due to the tangential velocity-slip at the porous surface, the reduction being more appreciable at higher frequencies of oscillation. When n is large, $\mid B \mid$ tends to take a common value for all values of A. On the assumption of 'no slip' at the boundary, this limiting value is found to be infinitely large. But when the boundary condition of tangential velocity-slip is adopted, the limiting value of $\mid B \mid$ is seen to be finite.

In order to show the variation in the phase lead of the fluctuating part of the skin friction, $\tan\theta$ has been plotted against $\log_{10} n$ in Fig. 2. The phase lead is

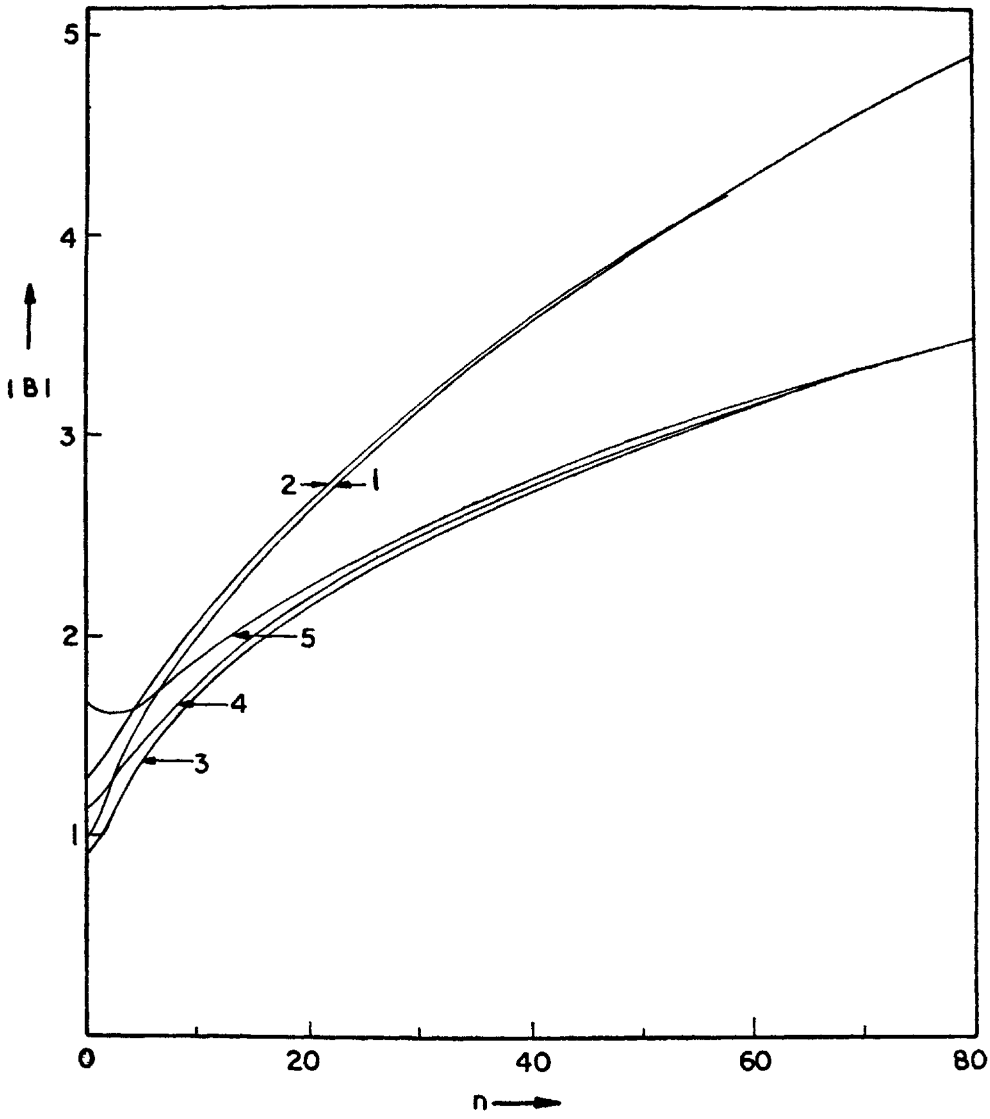

FIG. 1. Skin friction amplitude against frequency. (1) $A = 0$, (2) $A = 0.3$, when $\beta = 0 = K$; (3) $A = 0$, (4) $A = 0.3$, (5) $A = 0.9$, when $\beta = 10^{-1}$, $K = 10^{-4}$.

positive and is smaller when A is larger. This observation applies equally to the results calculated on the velocity-slip as well as the no slip boundary conditions. But as n increases, the phase lead tends to $\pi/4$ on the assumption of 'no slip', but it attains a maximum value and then decreases towards zero on the assumption of the velocity-slip boundary condition. The maximum value of the phase lead decreases as A increases.

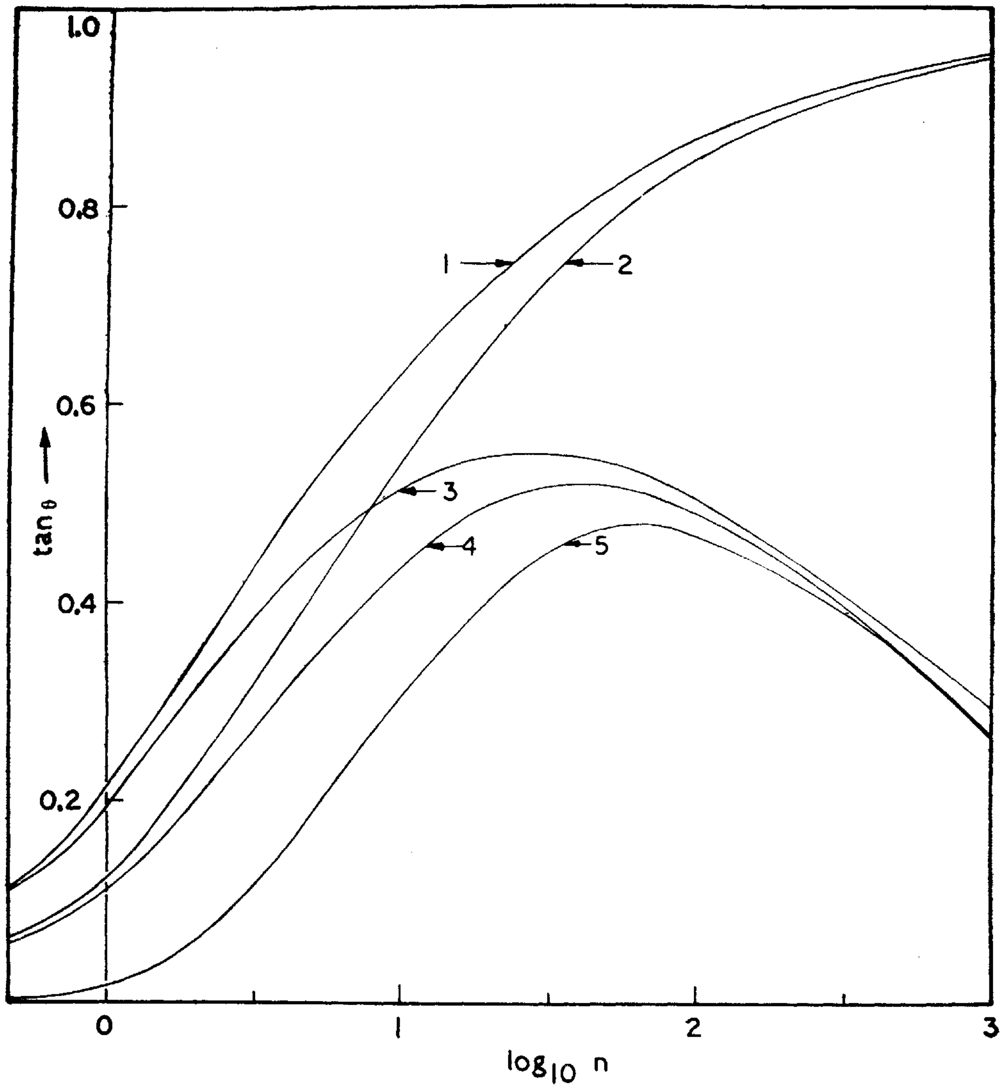

FIG. 2. Skin friction phase lead against frequency. (1) $A = 0$, (2) $A = 0.3$, when $\beta = 0 = K$; (3) $A = 0$, (4) $A = 0.3$, (5) $= 0.9$; when $\beta = 10^{-1}$, $K = 10^{-4}$.

The distribution of the transient stream-wise velocity in the boundary layer is shown against the dimensionless distance from the porous bed in Fig. 3 for the instants given by $nt = 0$, $\pi/4$, $\pi/2$ and $3\pi/4$, taking $A = 0.3$ and $n = 50$. The velocity is everywhere larger on the assumption of tangential velocity-slip than it is on the assumption of no-slip since the velocity slip at the boundary reduces the loss of energy by viscous dissipation. The magnitude of slip velocity at the permeable surface is not insignificant at any moment, but seen to be the largest at the beginning of each cycle of oscillation of the free-stream and suction velocities.

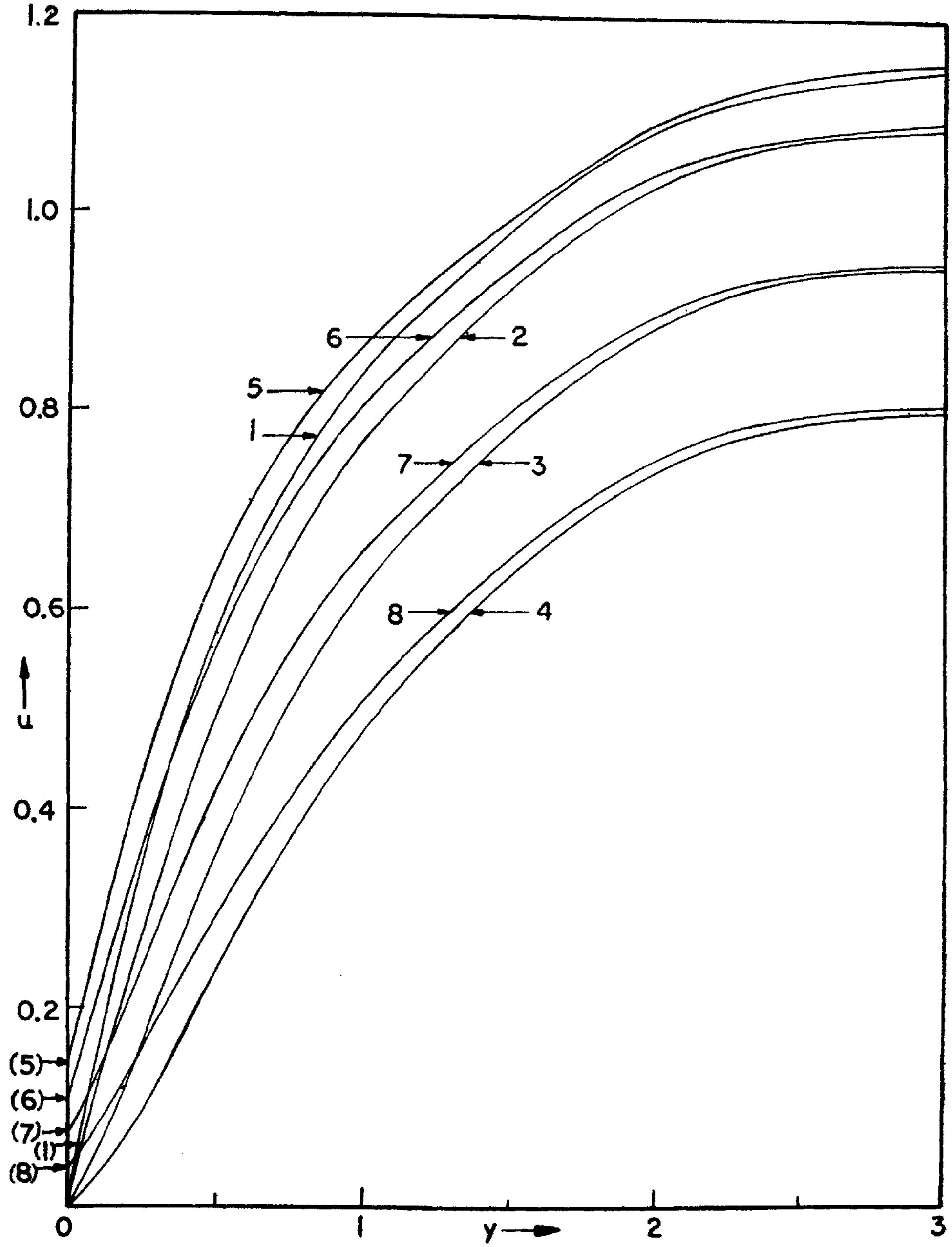

FIG. 3. Transient velocity profiles for $A = 0.3$, $\epsilon = 0.2$, $n = 50$. (1) at $nt = 0$, (2) at $nt = \pi/4$, (3) at $nt = \pi/2$, (4) at $nt = 3\pi/4$ when $\beta = 0 = K$; (5) at $nt = 0$, (6) at $nt = \pi/4$, (7) at $nt = \pi/2$, (8) at $nt = 3\pi/4$ when $\beta = 10^{-1}$, $K = 10^{-4}$.

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