AN INTEGRAL EQUATION INVOLVING ${}_4F_3$ IN THE KERNEL

by L. A. DIXIT, *Department of Mathematics, Govt. College of Engineering and Technology, Raipur* 492002, *M.P.*

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The solution of an integral equation involving ${}_4F_3$ is obtained by the application of the Rodrigues formula.

§1. The integral equation

$$\int_1^x k(x, t)\, g(t)\, dt = f(x), \; 1 < x < x_0 \qquad \ldots(1.1)$$

has been studied extensively in recent years by many workers (Joshi 1974, Rusia 1969, Singh 1967, Shrivastava 1965, Singh 1970) in the field with different functions in the kernel. In particular, Joshi (1974) has obtained the solution of (1.1) by taking Rice's polynomials as kernel. Dixit (1977) has extended the work of Joshi (1974) by using a generalized Rice's polynomial defined by Khandekar (1964) as kernel using a similar technique. He has also obtained corresponding Rodrigues formula for the kernel function.

In the present paper we solve an integral equation whose kernel is the hypergeometric function ${}_4F_3$. The solution of the integral equation embraces the known results as particular cases.

§2. The generalized Rice's polynomials are defined by Khandekar (1964), as

$$H_n^{(\alpha,\beta)}(\xi, p, \nu) = \frac{\Gamma(p)}{\Gamma(\xi)\,\Gamma(p-\xi)} \times \int_0^1 t^{\xi-1}(1-t)^{p-\xi-1}\, P_n^{(\alpha,\beta)}(1-2\nu t)\, dt,$$

$$\text{Re}\,(p) > \text{Re}\,(\xi) > 0. \qquad \ldots(2.1)$$

where

$P_n^{(\alpha,\beta)}(z)$ is the Jacobi polynomial.

He has also established the following

$$H_n^{(\alpha,\beta)}(\xi, p, \nu) = \frac{(1+\alpha)n}{(n)!}\, {}_3F_2\left[\begin{matrix} -n, \alpha+\beta+n+1, \xi; \\ p, \quad 1+\alpha\,; \end{matrix}\ \nu\right]. \qquad \ldots(2.2)$$

From (2.1) and (2.2) we easily obtain

$${}_3F_2\left[\begin{matrix} -n, \alpha+\beta+n+1, \xi; \\ p, \quad 1+\alpha \quad ; \end{matrix}\ \nu\right] = \frac{(n)!\,\Gamma(p)}{(1+\alpha)_n\,\Gamma(\xi)\,\Gamma(p-\xi)}$$

$$\times \int_0^1 t^{\xi-1}(1-t)^{p-\xi-1} P_n^{(\alpha,\beta)}(1-2\nu t)\,dt. \qquad \ldots(2.3)$$

Similarly we can establish

$${}_4F_3\left[\begin{matrix} -n, \alpha+\beta+n+1, \xi, \xi' \;; \\ p, \quad p', \quad 1+\alpha; \end{matrix}\ \nu\right] = \frac{(n)!\,\Gamma(p)}{(1+\alpha)_n\,\Gamma(\xi)\,\Gamma(p-\xi)}$$

$$\times \int_0^1 t^{\xi-1}(1-t)^{p-\xi-1} H_n^{(\alpha,\beta)}(\xi', p', \nu t)\,dt \qquad \ldots(2.4)$$

where $H_n^{(\alpha,\beta)}(\xi', p', \nu t)$ is defined in (2.1).

The Rodrigues formula for $H_n^{(\alpha,\beta)}(\xi', p', \nu t)$ obtained by Dixit (1977) is

$$H_n^{(\alpha,\beta)}(\xi', p', \nu t) = \frac{\nu^{-\alpha}}{A}\left(\frac{d}{d\nu}\right)^{\beta+n} \nu^{\alpha+\beta+n}$$

$$\times\, {}_2F_1\left[\begin{matrix} -n, \xi'; \\ p'; \end{matrix}\ \nu t\right] \qquad \ldots(2.5)$$

where $A = \dfrac{(n)!\,(1+\alpha)_{n+\beta}}{(1+\alpha)_n}$.

Equation (2.4) with the help of (2.5) reduces to the following form :

$${}_4F_3\left[\begin{matrix} -n, \alpha+\beta+n+1, \; \xi, \xi'; \\ p, \quad p', \quad 1+\alpha; \end{matrix}\ \nu\right]$$

(*equation continued on p.* 741)

$$= \frac{\Gamma(p)\, \nu^{-\alpha}}{\Gamma(\xi)\, \Gamma(p-\xi)\, (1+\alpha)_{n+\beta}} \cdot \left(\frac{d}{d\nu}\right)^{\beta+n} \nu^{\alpha+\beta+n}$$

$$\times \int_0^1 t^{\xi-1} (1-t)^{p-\xi-1} {}_2F_1 [-n, \xi'; p'; \nu t]\, dt. \qquad \ldots(2.6)$$

From Sneddon (1961, p. 99), we have the relation

$$\left(\frac{d}{dx}\right)^m \{x^{b+m-1} (1-x)^{a+m-b}\} = (b)_m\, x^{b-1} {}_2F_1 \left[\begin{matrix} b-a-m, b+m; \\ b; \end{matrix}\; x\right]. \qquad \ldots(2.7)$$

The above relation can be easily transformed into the following form

$$\left(\frac{d}{d\nu}\right)^{\xi'-p'} \{\nu^{\xi'-1} (1-\nu t)^n\} = (p')_{\xi'-p'} \cdot \nu^{p'-1} {}_2F_1 \left[\begin{matrix} -n, \xi'; \\ p'; \end{matrix}\; \nu t\right]. \qquad \ldots(2.8)$$

Substituting the value of ${}_2F_1 [-n, \xi'; p', \nu t]$ from (2.8) in (2.6), we get after simplification

$${}_4F_3 \left[\begin{matrix} -n, \alpha+\beta+n+1, \xi, \xi'; \\ p, \quad p', \quad 1+\alpha; \end{matrix}\; \nu\right] = \frac{\Gamma(p')\, \Gamma(1+\alpha)\, \nu^{-\alpha}}{\Gamma(\xi')\, \Gamma(\alpha+\beta+n+1)}$$

$$\times \left(\frac{d}{d\nu}\right)^{\beta+n} \nu^{\alpha+\beta+n-p'+1} \cdot \left(\frac{d}{d\nu}\right)^{\xi'-p'} \nu^{\xi'-1} {}_2F_1 [-n, \xi; p; \nu] \qquad \ldots(2.9)$$

Using eqn. (2.7), eqn. (2.9) can be put in the form :

$${}_4F_3 \left[\begin{matrix} -n, \alpha+\beta+n+1, \xi, \xi'; \\ p, \quad p', \quad 1+\alpha; \end{matrix}\; \frac{x}{t}\right]$$

$$= \frac{\Gamma(p)\, \Gamma(p')\, \Gamma(\alpha+1)}{\Gamma(\xi)\, \Gamma(\xi')\, \Gamma(\alpha+\beta+n+1)}\, t^{-n} x^{-\alpha}$$

$$\times \left(\frac{d}{dx}\right)^{\beta+n} x^{\alpha+\beta+n-p'+1} \left(\frac{d}{dx}\right)^{\xi'-p'} x^{\xi'-p} \left(\frac{d}{dx}\right)^{\xi-p} \{x^{\xi-1} (t-x)^n\}. \qquad \ldots(2.10)$$

which is the required Rodrigue's formula for ${}_4F_3$.

§3. *Theorem* — If

(i) $n \geqslant 1$, the function $f(x)$ be absolutely continuous on $(1, x_0)$ for some $x_0 > 1$;

(ii) $\alpha, \beta, p, p', \xi, \xi'$ are all non-negative integers; and

(iii) $f(1) = 0$.

Then, the integral equation

$$\int_1^x {}_4F_3\left[\begin{matrix} -n, \alpha+\beta+n+1, & \xi, \xi'; \\ & & \dfrac{x}{t} \\ p, \quad p', & 1+\alpha; \end{matrix}\right] g(t)\,dt = f(x),$$

$$\text{for } 1 < x < x_0 \qquad \ldots(3.1)$$

has the solution

$$g(x) = cx^n \left(\frac{d}{dx}\right)^{n+1} \{x^{-(\xi-1)} I_{\xi-p}[M_1(x)]\} \qquad \ldots(3.2)$$

where $C = \dfrac{(-1)^n B}{(n)!}$.

PROOF : Substituting the value of ${}_4F_3\left[\begin{matrix} -n, \alpha+\beta+n+1, & \xi, \xi'; \\ & & \dfrac{x}{t} \\ p, \quad p', & 1+\alpha; \end{matrix}\right]$

from (2.10) in (3.1), and interchanging the operators of integration and differentiation, since the interchange is justified, we have

$$\left(\frac{d}{dx}\right)^{\beta+n} \int_1^x x^{\alpha+\beta+n-p'+1} \left(\frac{d}{dx}\right)^{\xi'-p'} x^{\xi'-p}$$

$$\times \left(\frac{d}{dx}\right)^{\xi-p} \{x^{\xi-1}(t-x)^n\}.t^{-n} g(t)\,dt = Bx^{\alpha} f(x) \qquad \ldots(3.3)$$

where $B = \dfrac{\Gamma(\xi)\,\Gamma(\xi')\,\Gamma(\alpha+\beta+n+1)}{\Gamma(p)\,\Gamma(p')\,\Gamma(1+\alpha)}$.

The successive integration $(\beta + n)$ times, and change of operators of integration and differentiation gives us

$$\left(\frac{d}{dx}\right)^{\xi'-p'} \int_1^x x^{\xi'-p} \left(\frac{d}{dx}\right)^{\xi-p} \{x^{\xi-1}(t-x)^n\}.t^{-n} g(t)\,dt$$

(*equation continued on p.* 743)

$$= Bx^{-(\alpha+\lambda-p'+1)} - \frac{1}{(\lambda-1)!}\int_1^x (x - v)^{\lambda-1} F(v)\, dv. \qquad \ldots(3.4)$$

where

$$\lambda = \beta + n, \text{ and } F(v) = v^{\alpha} f(v).$$

We define the operator

$$I_\lambda\,[F(x)] = \frac{1}{(\lambda-1)!}\int_1^x (x - v)^{\lambda-1} F(v)\, dv.$$

$$\therefore \left(\frac{d}{dx}\right)^{\lambda} I_\lambda\,[F(x)] = F(x)$$

and

$$I_\lambda\,[F(x)] = 0, \text{ for } x = 1.$$

The above equation takes the form

$$\left(\frac{d}{dx}\right)^{\xi'-p'} \int_1^x x^{\xi'-p}\left(\frac{d}{dx}\right)^{\xi-p} \{x^{\xi-1}(t - x)^n\}\,.\,t^{-n} g(t)\, dt = BM(x) \qquad \ldots(3.5)$$

where $\qquad M(x) = x^{-(\alpha+\lambda-p'+1)}\,.\,I_\lambda\,[F(x)].$

Successive integration $(\xi' - p')$ times, yields

$$\int_1^x \left(\frac{d}{dx}\right)^{\xi-p} \{x^{\xi-1}(t - x)^n\}\,.\,t^{-n} g(t)\, dt = \frac{x^{p-\xi'}}{(\xi' - p' - 1)!\, B}$$

$$\times \int_1^x (x - u)^{\xi'-p'-1} M(u)\, du.$$

Taking $\xi' - p' = \lambda'$ and changing the order of integration and differentiation, which is justified, we get

$$\left(\frac{d}{dx}\right)^{\xi-p} \int_1^x x^{\xi-1}(t - x)^n\,.\,t^{-n} g(t)\, dt = Bx^{p-\xi'} I_{\lambda'}\,[F_1(x)] \qquad \ldots(3.6)$$

where $\qquad I_{\lambda'}\,[F_1(x)] = \dfrac{1}{(\lambda'-1)!}\displaystyle\int_1^x (x - u)^{\lambda'-1} M(u)\, du.$

$$\therefore \left(\frac{d}{dx}\right)^{\lambda'} I_{\lambda'}[F_1(x)] = F_1(x)$$

and $\qquad I_{\lambda'}[F_1(x)] = 0, \text{ for } x = 1.$

Equation (3.6) can be put in the form

$$\left(\frac{d}{dx}\right)^{\xi-p} \int_1^x x^{\xi-1}(t-x)^n . t^{-n} g(t)\, dt = BM_1(x) \qquad \text{...(3.7)}$$

where $\qquad M_1(x) = x^{p-\xi'} I_{\lambda'}[F_1(x)].$

Taking repeated integration $(\xi - p)$ times, we get

$$\int_1^x (t-x)^n . t^{-n} g(t)\, dt = Bx^{-(\xi-1)} I_{\xi-p}[M_1(x)] \qquad \text{...(3.8)}$$

where $\qquad I_{\xi-p}[M_1(x)] = \dfrac{1}{(\xi-p-1)!}\displaystyle\int_1^x (x-y)^{\xi-p-1} . M_1(y)\, dy.$

Finally differentiating $n + 1$ times with respect to x, we obtain the required result (3.2).

Putting $\xi' = p' = 1$ and replacing ξ by $\xi + p$ in (3.1) and (3.2), we get the result obtained by Dixit (1977).

Furthermore, if we put $\alpha = \beta = 0$ the result reduces to that of Joshi (1974).

§4. The integral equation

$$\int_1^x \left(\frac{x}{t}\right)^{\mu} \times {}_4F_3\left[\begin{matrix} -n, \alpha+\beta+n+1, \xi, \xi'; \\ p, \quad p', \quad 1+\alpha; \end{matrix}\ \frac{x}{t}\right] g_1(t)\, dt$$

$$= f_1(x) \text{ for } 1 < x < x_0 \qquad \text{...(4.1)}$$

is reduced to (3.1) if $\mu = 0$. However, if we write $t^{-\mu} g_1(t) = g(t)$ and $t^{-\mu} f_1(t) = f(t)$, eqn. (4.1) is again reduced to (3.1) hence we get the solution of (4.1) as

$$g_1(x) = Cx^{n+\mu}\left(\frac{d}{dx}\right)^{n+1} \{x^{-(\xi-1)} . I_{\xi-p}[M_1(x)]\}. \qquad \text{...(4.2)}$$

where the symbols have their usual meanings.

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