ON THE CONTINUITY OF GROUP OPERATIONS

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A group $(G, \cdot)$ endowed with a topology τ is called a semitopological (quasitopological) group iff the group operation $(x, y) \to x \cdot y$ is continuous in each variable separately (both variables jointly). It is shown in this paper that a T_1 quasitopological group $(G, \tau, \cdot)$ is a topological group if it satisfies any one of the following conditions : (i) τ is compact; (ii) τ is countably compact and the identity element is a G_δ; (iii) τ is a Lindelöf P-space; (iv) τ is sequential and sequentially compact. In fact, under condition (ii), the quasitopological group becomes a compact metric topological group.

In this paper topological conditions which imply the joint continuity of the group operation $(x, y) \to x \cdot y$ and the continuity of the operation $x \to x^{-1}$ in a group G (endowed with a topology τ) are studied. Here x and y are elements of G. Throughout the discussion it is assumed that (G, τ) is a semitopological group in the sense that the group operation $(x, y) \to x \cdot y$ is continuous in both variables separately (Hussain 1965, 1966). A group G endowed with a topology τ is a quasitopological group if and only if the group operation $(x, y) \to x \cdot y$ is continuous in both variables jointly. Conditions for a semitopological group to be a quasitopological group and a quasitopological group to be a topological group are obtained. N and e denote the natural numbers and the identity element of G respectively.

Every semitopological group (G, τ) can be imbedded in $C = C(G, G)$, the set of all τ-continuous functions of G into itself. If $\eta_r(G)$ denotes the set of all right translations $r_a : x \to xa$, then G can be identified with $\eta_r(G) \subset C$. If, further, G is a semitopological group with reference to τ', a topology on G distinct from τ, then G can be imbedded in $D = C \cap C'$ where $C' = C'(G, G)$, the set of all τ'-continuous functions of G into itself. Let σ be the topology on D generated by the sets of the form $W(K, U) = \{f \mid f \in C, f(K) \subset U\} \cap D$ where K is τ'-closed and U is τ-open. Interchanging τ and τ' and replacing C by C', we define similarly the topology σ' on D. Let ρ be the relative topology on G of the product of τ on G^G. Let us define ρ' in a similar fashion. Let μ and μ' denote the relative topologies of σ and σ' respectively on G. If τ and τ' are T_1, then $\rho \subset \mu$ and $\rho' \subset \mu'$.

In our results the topology τ' in the previous paragraph is not arbitrary. Whenever (G, τ) is a quasitopological (semitopological) group, let us define

$\tau' = \{U \mid U^{-1} \in \tau\}$. Clearly τ' is a topology and (G, τ') is a quasitopological (semitopological) group in its own right. Furthermore, $x \to x^{-1}$ is a homeomorphism of (G, τ) to (G, τ'). τ' is called the 'conjugate topology' of τ. We call U' 'conjugate open' if $U' \in \tau'$. The τ'-closure of a subset A is called the 'conjugate closure' of A and is written as 'cocl (A)'. It is clear that this relation between τ and τ' is symmetrical.

The following proposition is easily proved.

Proposition 1 — Let (G, τ) be a quasitopological group. Then,

(i) $\operatorname{cocl}(A) = \bigcap \{AU \mid U \in \tau \text{ and } e \in U\}$ for any subset $A \subset G$;

(ii) if $e \in U \in \tau$ then $e \in V \subset \operatorname{cocl}(V) \subset U$ for some $V \in \tau$;

(iii) τ is T_1 if and only if for every pair of distinct points $x, y \in G$, there exist disjoint sets U and U' which are open and conjugate open respectively such that $x \in U$ and $y \in U'$.

Now let us give a sufficient condition for a semitopological group to be a quasitopological group.

Theorem 2 — If (G, τ) is a semitopological group which satisfies the property (ii) of Proposition 1, then the map $f : (G \times G, \tau \times \mu) \to (G, \tau)$ given by $(x, y) \to x \cdot y$ is continuous in both variables jointly. In particular, (G, τ) is a quasitopological group if $\tau = \mu$.

PROOF: If U is a τ-open neighbourhood (= *nbd*, hereinafter) of $x \cdot y$, then $x \to x \cdot y = r_y(x)$ is continuous in τ so that there exists an open *nbd* L of x such that $r_y(L) = LY \subset U$. Since G satisfies property (ii) of Proposition 1, there exists a τ-open set K such that $x \in K \subset \operatorname{cocl}(K) \subset L$. This implies $\operatorname{cocl}(K) \cdot y \subset U$ i.e., $r_y \in W(\operatorname{cocl}(K), U)$. Hence, by identifying $\eta_r(G)$ with G, we get $y \in W(\operatorname{cocl}(K), U) \cap G$. Now if $z \in W(\operatorname{cocl}(K), U) \cap G$, then $\operatorname{cocl}(K)\, z \subset U$, so that $Kz \subset (\operatorname{cocl}(K))\, z \subset U$ and hence $K \cdot (W(\operatorname{cocl}(K), U) \cap G) \subset U$. Hence the result.

Remarks: It is to be noticed that when $\tau = \mu$, then $\tau' = \mu'$ and (G, τ') turns out to be a quasitopological group in its own right. It can be proved that the map $x \to x^{-1}$ is a homeomorphism from (G, μ) onto (G, μ').

Proposition 3 — The map $x \to x^{-1}$ under the conditions of Theorem 2 is a homeomorphism of $(G, \mu) \to (G, \mu')$.

PROOF: Suppose $r_x = x \in W(K, U)$ when K is τ'-closed and U is τ-open. Hence $r_x(K) = Kx \subset U$ so that $G - U \subset (G - K)\, x$ and hence $x^{-1} \in W(G - U, G - K)$ where $G - K$ is τ'-open and $G - U$ is τ-closed.

Theorem 4 — Let (G, τ) be a T_1 quasitopological group. Then the map $x \to x^{-1}$ is continuous with respect to τ (i.e., (G, τ) is a topological group) if any one of the following conditions is satisfied :

(i) τ is compact.

(ii) e admits a conjugate open *nbd* which is τ-compact.

(iii) Every τ-open cover of G has a τ-open refinement which is conjugate locally finite.

(iv) τ is countably compact and $\{e\}$ is a G_δ with respect to τ.

(v) e admits a conjugate open *nbd* which is τ-countably compact and $\{e\}$ is a G_δ with respect to τ.

(vi) Every countable τ-open cover of G has a τ-open refinement which is conjugate locally finite and $\{e\}$ is a G_δ with respect to τ.

(vii) τ is Lindelöf and a P-space [refer Gillman and Jerrison (1960) for definition].

(viii) e admits a conjugate open *nbd* which is τ-Lindelöf and τ is a P-space.

(ix) Every τ-open cover of G has a τ-open refinement which is conjugate locally countable and τ is a P-space.

PROOF : First let us prove the theorem to be true for (ii) and (v); then this would imply that the theorem is true for (i) and (iv) respectively. The proof for (viii) and (ix) is similar to (ii) and (iii); also the validity of the theorem (viii) implies the validity of the theorem for (vii).

Assume that (ii) is satisfied. Let $x \in G$ and A be a conjugate open *nbd* of x which is compact. Then there exists a conjugate open set Q such that $x \in Q \subset \text{cl } Q \subset A$. This is possible because translations are homeomorphisms and Proposition 1 (ii) holds. Hence cl (Q) is a conjugate *nbd* of x which is compact. Let P be any open set containing x and take $X = Q - P$. Then cl (X) is compact and $x \notin$ cl (X). Further by virtue of Proposition 1 (iii) and because cl (X) is compact, cl (X) is co-closed. Let $V = G - \text{cl } (X)$. Then V is a conjugate open *nbd* of x and $V \cap X = \phi$. If $U = Q \cap V$, then U is a conjugate open set containing x and moreover if $y \in U$, then $y \in Q$ but $y \notin X$ so that $y \in P$. Thus $x \in U \subset P$. Hence $\tau \subset \tau'$, the conjugate of τ. Symmetrically $\tau' \subset \tau$ and hence the result.

Assume that (iii) is satisfied. Let $x \in U \in \tau$. By Proposition 1 (iii), for each point $y \in G - U$, there exist an open set U_y and a conjugate open set V_y such that $y \in U_y$, $x \in V_y$ and $U_y \cap V_y = \phi$. Let $\alpha = \{U, \{U_y \mid y \in G - U\}\}$. α is an open cover of G. Hence α has an open refinement β which is conjugate locally finite. Let M be a conjugate open set with $x \in M$ which meets a finite subcollection β_1 of β. Let $\{W_i \mid i = 1, 2, 3, \ldots, n\} \subset \beta_1$ such that no W_i is contained in U for $i = 1, 2, 3, \ldots, n$. (If there are no such sets then $x \in M \subset U$). There are points $y_1, y_2, y_3, \ldots, y_n$ in $G - U$ such that $W_i \subset U_{y_i}$ $(i \equiv 1, 2, 3, \ldots, n)$. Let

$$P = \cap \{M \cap V_{y_i} \mid i = 1, 2, 3 \ldots, n\}.$$

Then P is conjugate open and $x \in P \subset U$. Hence it follows that $\tau \subset \tau'$, the conjugate of τ. Symmetrically $\tau' \subset \tau$ and hence the result.

Assume that (v) is satisfied. Let $x \in G$ and A be a conjugate open *nbd* of x which is countably compact. Then there exists a conjugate open set Q such that $x \in Q \subset \text{cl}(Q) \subset A$. Then cl (Q) is a conjugate *nbd* of x which is countably compact. Let P be any open set containing x and take $X = Q - P$. Now cl (X) is countably compact and $x \notin X$. Indeed $x \notin \text{cl}(X)$. By Proposition 1 (iii),

$$\{x\} = \cap \{\text{cl}(V_n) \mid n \in N\}$$

where each V_n is a conjugate open *nbd* of x. Thus $\{G - \text{cl}(V_n) \mid n \in N\}$ is a countable open cover of cl (X). It has a finite subcover $\{G - \text{cl}(V_{n_i}) \mid i = 1, 2, 3 \ldots, k\}$ and $V = \cap \{V_{n_i} \mid i = 1, 2, 3 \ldots, k\}$ is a conjugate open *nbd* of x with $V \cap X = \phi$. If $U = Q \cap V$, then U is a conjugate open set and $x \in U \subset P$. Hence $\tau \subset \tau'$. Symmetrically $\tau' \subset \tau$ and hence the result.

Assume that (vi) is satisfied. Since each point is a G_δ, by Proposition 1 (ii) each point can be expressed as a countable intersection of τ-closures of conjugate open *nbds*. Let $x \in U \in \tau$. Now $\{x\} = \cap \{\text{cl}(V_n) \mid n \in N\}$ where each V_n is conjugate open. Then $\alpha = \{U, \{G - \text{cl}(V_n) \mid n \in N\}\}$ is a countable open cover of G and now the proof proceeds in the same line as that of (iii) above.

It is well known that every locally countably compact topological group for which $\{e\}$ is a G_δ is a locally compact metrizable topological group (Hewitt and Ross 1963, page 70). Hence the condition (v) of Theorem 4 gives the following result.

Theorem 5 — Every quasitopological group (G, τ) for which $\{e\}$ is G_δ is a locally compact metrizable topological group if there exist an open *nbd* V of e such that V^{-1} is countably compact.

Sequential spaces have been studied extensively by Franklin (1965, 1967). A subset of a topological space is sequentially closed if and only if no sequence in it converges to a point not in it. A topological space X is called sequential if and only if every sequentially closed subset of X is closed.

Theorem 6 — Let (G, τ) be a quasitopological group whose topology is sequential and T_1. Then (G, τ) is a topological group if either of the following conditions is satisfied :

(i) τ is sequentially compact.

(ii) e admits a conjugate open *nbd* which is τ-sequentially compact.

PROOF : If the theorem is true for (ii), then it is true for (i). Hence let us prove the theorem for (ii). Let us prove first that every co-sequentially compact subset of G is closed. If τ is sequential it is clear that τ' is also sequential. Let K be a co-sequentially compact subset of G and $\{x_n\}$ be a sequence in K such that x_n converges to x_0 with respect to τ. Clearly $\{x_0\} \cup \{x_n \mid n \in N\}$ is sequentially closed in the conjugate topology and hence co-closed, since x_0 is the only possible co-accumulation point of $\{x_n\}$ (by Proposition 1 (iii)). If $\{x_n \mid n \in N\}$ is infinite then $x_0 \in K$. If not, then $x_n = x_0$ for $n \geqq n_0$. Thus every sequence in K converges to a point in K only. It follows K is sequentially closed and hence closed. Thus every co-sequentially compact set is τ-closed. Now suppose $x \in G$ and A be a *nbd* of x which is co-sequentially compact. By Proposition 1 (ii), there is an open set Q such that $x \in Q \subset \text{cocl}\,(Q) \subset A$. Hence $\text{cocl}\,(Q)$ is a *nbd* of x which is co-sequentially compact. Let P be any conjugate open set containing x. Let $B = Q - P$. Then $\text{cocl}\,(B) \subset \text{cocl}\,(Q)$ and hence $\text{cocl}\,(B)$ is co-sequentially compact. Therefore $\text{cocl}\,(B)$ is τ-closed. Further $x \notin \text{cocl}(B)$. Take $V = G - \text{cocl}\,(B)$. Clearly $V \cap B = \phi$; V is open and $x \in V$. Now $Q \cap V$ is an open set such that $x \in Q \cap V \subset P$. Hence $\tau' \subset \tau$. By symmetry, the result follows.

It was proved by Rajagopalan and Soundararajan (1968) that a locally compact group G which is sequential is metrizable. The Theorem 6 above together with Franklin's result that in a sequential space countable compactness and sequential compactness are equivalent, therefore, now gives the following result.

Theorem 7 — If the underlying space of a quasitopological group is sequential and T_1 such that the identity admits an open *nbd* U with U^{-1} being compact, then the group is a locally compact metrizable topological group.

In view of Theorem 4 (iv) and Theorems 6 and 7 two very attractive conjectures would be :

Conjecture I — If (G, τ) is a T_1 quasitopological group, then the map $x \to x^{-1}$ is continuous if τ is psuedo-compact and $\{e\}$ is G_δ.

Conjecture II — If (G, τ) is a T_1 sequential quasitopological group admitting a symmetric sequentially compact *nbd* of the identity, then (G, τ) is a locally compact metrizable topological group.

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