POLYNOMIAL STRUCTURE ON ALMOST PARACONTACT MANIFOLD

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In this paper a quartic structure $f^4 - (1 + \lambda\eta(N)) f^2 + \eta(N) I = 0$ has been derived from almost paracontact manifold (Sato 1976), from which para f-structure (Singh and Vohra 1972; Sinha 1977) and para (f, g, u, v, λ)-structure (Sinha 1977) have been obtained and studied proceeding on the lines of the paper of Goldberg and Yano (1970).

1. INTRODUCTION

Let M be a $n + 1$ dimensional almost paracontact manifold with fundamental tensor field ϕ of type (1, 1), fundamental vector field E and fundamental 1-form η. Consider an n-dimensional manifold P embedded in M with embedding $b : P \to M$ and assumed that for each $p \in P$ the vector field E at $b(p)$ does not belong to the tangent hyperplane of the hypersurface. This means that the fundamental vector field of M can be taken as the "affine normal" to the hypersurface, we therefore have (Goldberg and Yano 1970)

$$\phi BX = BJX + \alpha(X)\, E, \phi E = 0 \qquad ...(1.1)$$

where B is the differential map of b. If $\alpha \neq 0$, we call $b\,(P)$ a non-invariant hypersurface of M. The structure J induced on P by ϕ is almost product.

2. QUARTIC STRUCTURE

Let $b(P)$ be a non-invariant hypersurface of the almost paracontact $M(\phi, E, \eta)$ (Sato 1976). We wish to choose an affine normal N on $b(P)$ in such a way that the vector field ϕN is always tangent to the hypersurface, that is

$$\phi N = -\, BU \qquad ...(2.1)$$

for some vector field U on P.

Since the vector field N is not tangent to the hypersurface it can be represented as

$$N = \frac{1}{\lambda}\,(-\, BX + E)$$

for a certain vector field X and scalar field $\lambda \neq 0$. We have

$$\phi N = -\frac{1}{\lambda}(BJX + \alpha(X)\,E)$$

by virtue of (1.1). Thus for (2.1) to hold, we must have $\alpha(X) = 0$. We therefore, assume that a global vector field V exists which satisfies this equation, that is $\alpha(V) = 0$. Putting $X = V$ in the above equation and using (2.1), we have $U = (1/\lambda)\,JV$, and

$$E = BV + \lambda N,\ \lambda \neq 0. \qquad \text{...(2.2)}$$

Hence by setting $\beta(X) = -\alpha(JX)$, we have

$$\alpha(V) = 0,\ \ \beta(U) = 0 \qquad \text{...(2.3)}$$

$$JV = \lambda U,\ \ JU = \frac{1}{\lambda}\,V. \qquad \text{...(2.4)}$$

From (1.1) it is easily seen that $\beta(X) = \eta(BX)$. Operating (2.2) by η and using the above equations, we have

$$\lambda\eta(N) + \beta(V) = 1.$$

From (1.1) and (2.2)

$$\phi BX = B(J + \alpha \otimes V)\,X + \lambda\alpha(X)\,N,$$

that is

$$\phi BX = BJ'X - \alpha'(X)\,N, \qquad \text{...(2.5)}$$

where

$$J' = J + \alpha \otimes V,\ \ \alpha' = -\lambda\alpha. \qquad \text{...(2.6)}$$

Thus $\quad J'^2 = I - \alpha' \otimes U - \beta \otimes V.$

Theorem 2.1 — Let $P(J, \alpha)$ be a non-invariant hypersurface of the almost paracontact manifold $M(\phi, E, \eta)$. If there is a global vector field V on P such that $\alpha(V) = 0$ then, the tensor fields J', α', β, U, V on P satisfy the relations (Sinha 1977)

$$\left.\begin{aligned} &J'^2 = I - \alpha' \otimes U - \beta \otimes V \\ &J'U = \eta(N)\,V,\ \ J'V = \lambda U \\ &\alpha' \circ J' = \lambda\beta,\ \ \beta \circ J' = \eta(N)\,\alpha' \\ &\alpha'(U) = 1 - \lambda\eta(N),\ \ \alpha'(V) = 0 \\ &\beta(U) = 0,\ \ \beta(V) = 1 - \lambda\eta(N). \end{aligned}\right\} \qquad \text{...(2.7)}$$

Corollary 2.1 — If the vector fields E and N are distinct affine normals, then the structure on P is quartic structure, i.e.

$$f^4 - (1 + \lambda\eta(N))\, f^2 + \lambda\eta(N)\, I = 0,$$

where $f = J'$.

The left side of this equation can be factored, that is $(f^2 - \lambda\eta(N) I)(f^2 - I) = 0$. There may be three cases namely, $\lambda = 1$, $\eta(N) \neq 1$; $\lambda\eta(N) = 1$; and $\eta(N) = \lambda$.

Case I : $\lambda = 1$ and $\eta(N) \neq 1$.

Corollary 2.2 — Let P be a non-invariant hypersurface of an almost paracontact manifold (Sato 1976). Then if $\lambda = 1$ and $\eta(N) \neq 1$, P is not manifold with para f-structure (Singh and Vohra 1972, Sinha 1977) that is

$$\left.\begin{aligned} & f^2 = I - (1 - \eta(N))\,[\tilde{\alpha} \otimes \tilde{U} + \tilde{\beta} \otimes \tilde{V}] \\ & \tilde{\alpha}(\tilde{U}) = 1,\ \tilde{\alpha}(\tilde{V}) = 0 \\ & \tilde{\beta}(\tilde{U}) = 0,\ \tilde{\beta}(\tilde{V}) = 1 \\ & f\tilde{U} = \eta(N)\,\tilde{V},\ f\tilde{V} = \tilde{U} \\ & \tilde{\alpha} \circ f = \tilde{\beta},\ \ \tilde{\beta} \circ f = \eta(N)\,\tilde{\alpha} \end{aligned}\right\} \qquad \ldots(2.8)$$

where

$$\tilde{U} = \frac{1}{1 - \eta(N)}\, U,\ \ \tilde{V} = \frac{1}{1 - \eta(N)}\, V,\ \tilde{\alpha} = \alpha' \text{ and } \tilde{\beta} = \beta.$$

PROOF : Putting $\tilde{U} = \frac{1}{1 - \eta(N)}\, U$, $\tilde{V} = \frac{1}{1 - \eta(N)}\, V$, $\tilde{\alpha} = \alpha'$ and $\tilde{\beta} = \beta$ in (2.7), we get (2.8).

Corollary 2.3 — Let P be a non-invariant hypersurface of an almost paracontact manifold, then if $\lambda = 1$ and $\eta(N) = 0$, P is a manifold with para f-structure, that is

$$\left.\begin{aligned} & f^2 = I - \alpha \otimes U - \beta \otimes V \\ & fU = 0,\ fV = U \\ & \alpha \circ f = \beta,\ \ \beta \circ f = 0 \\ & \beta(U) = 0,\ \beta(V) = 1 \\ & \alpha(U) = 1,\ \alpha(V) = 0. \end{aligned}\right\} \qquad \ldots(2.9)$$

PROOF : Putting $\lambda = 1$ and $\eta(N) = 0$ in (2.7), we have (2.9).

Case II : $\lambda\eta(N) = 1$.

Corollary 2.4 — Let P be a non-invariant hypersurface of an almost paracontact manifold. Then if $\lambda\eta(N) = 1$, we have

$$(f^2 - I)^2 = 0$$

and

$$\left.\begin{aligned} &f^2 = I - \alpha \otimes U' - \beta \otimes V \\ &fU' = - V,\ fV = - U' \\ &\alpha \circ f = - \beta,\ \beta \circ f = - \alpha \\ &\alpha(U') = 0,\ \alpha(V) = 0 \\ &\beta(U') = 0,\ \beta(V) = 0. \end{aligned}\right\} \qquad \ldots(2.10)$$

where $U' = - \lambda U$ and $\alpha' = - \lambda\alpha$.

PROOF : Putting $\lambda = \dfrac{1}{\eta(N)}$ in (2.7), we get (2.10).

Case III : $\eta(N) = \lambda$ If $\eta(N) = \lambda$, then quartic structure will be

$$f^4 - (1 + \lambda^2) f^2 + \lambda^2 = 0.$$

Corollary 2.5 — Let P be a non-invariant hypersurface of an almost paracontact manifold M. Then if $\eta(N) = \lambda$, P is a manifold with para $(f, U, V, \alpha', \beta, \lambda)$-structure (Sinha 1977, Yano and Okumura 1970) that is,

$$\left.\begin{aligned} &f^2 = I - \alpha' \otimes U - \beta \otimes V \\ &fU = \lambda V,\ fV = \lambda U \\ &\alpha' \circ f = \lambda\beta,\ \beta \circ f = \lambda\alpha' \\ &\alpha'(U) = 1 - \lambda^2,\ \alpha'(V) = 0 \\ &\beta(U) = 0,\ \beta(V) = 1 - \lambda^2. \end{aligned}\right\} \qquad \ldots(2.11)$$

PROOF : Putting $\lambda = \eta(N)$ in (2.7), we get (2.11).

3. HYPERSURFACES OF ALMOST PARACONTACT MANIFOLD WITH (ϕ, E, η)-CONNEXIONS

If almost paracontact manifold satisfying (ϕ, E, η) connexion, then $\nabla\phi = 0$, $\nabla E = 0$, and $\nabla\eta = 0$, where ∇ denotes covariant differentiation with respect to a symmetric affine connection on M. Since $\phi^2 = I - \eta \otimes E$ the vector field E is also parallel with respect to ∇. Denoting by D the induced connexion on the hypersurface P with respect to the affine normal N, the equation of Gauss and Weingarten are

$$(D_X B)(Y) = h(X, Y) N \qquad \ldots(3.1)$$

and

$$D_X N = - BHX + w(X) N \qquad \ldots(3.2)$$

respectively, where h and H are the second fundamental tensors of type (0, 2) and (1, 1) respectively of P with respect to the affine normal N, the tensor h being symmetric and w is a 1-form on P.

Differentiating (2.1), (2.2), (2.5) and $\eta(BY) = \beta(Y)$ covariantly and using (3.1), (3.2), we get (Goldberg and Yano 1970, Singh and Vohra 1972)

$$\left.\begin{aligned}
&(D_X J')(Y) = -\alpha'(Y)\,H(X) - h(X, Y)\,U \\
&D_X V = \lambda HX, \quad D_X U = J'HX + w(X)\,U \\
&(D_X\beta)(Y) = h(X, Y)\,\eta(N) \\
&(D_X\alpha')(Y) = h(X, J'Y) - w(X)\,\alpha'(Y) \\
&h(X, V) = -X.\lambda - \lambda w(X) \\
&h(X, U) = -\alpha'(HX).
\end{aligned}\right\} \qquad \ldots(3.3)$$

Theorem 3.1 — Let P with para f-structure be non-invariant hypersurface of an almost paracontact manifold with (ϕ, E, η) connexion. Then with respect to the induced connexion D on P, the quartic structure on P satisfies the relations

$$\left.\begin{aligned}
&(D_X f)(Y) = \alpha(Y)\,HX - h(X, Y)\,U \\
&D_X V = HX, \quad D_X U = fHX + w(X)\,U \\
&D_X\beta = 0, \quad (D_X\alpha)(Y) = -h(X, fY) - w(X)\,\alpha(Y) \\
&h(X, V) = -w(X), \quad h(X, U) = \alpha(HX) \\
&\beta(HX) = 0.
\end{aligned}\right\} \qquad \ldots(3.4)$$

PROOF : Putting $f = J'$, $\alpha = -\alpha'$, $\lambda = 1$ and $\eta(N) = 0$ in (3.3), we get (3.4). The last formula follows by differentiating $\eta(N) = 0$.

If for every vector field X on P, $HX = 0$, then by Weingarten's equation, $D_X N$ and N are proportional. Hence the affine normals are parallel along the hypersurface. In this case P is said to be totally geodesic. We are now able to deduce the following theorem.

Theorem 3.4 — If the hypersurface P with para f-structure is affinely umbilical in an almost paracontact manifold, then it is totally flat.

PROOF : Since P is affinely umbilical, $H = \mu I$. Hence $0 = \beta(HX) = \mu\beta(X)$. But $\beta(V) = 1$, so μ must vanish.

Theorem 3.5 — Let P be a non-invariant hypersurface with para f-structure of an almost paracontact manifold with (ϕ, E, η)-connexion. Then, if the linear transformation field f is parallel,

$$h = \mu\alpha \otimes \alpha, \quad H = \mu\alpha \otimes U, \quad w = 0 \qquad \ldots(3.5)$$

for some function μ depending on U and h.

PROOF : To see this, we first observe that from (3.4), if the linear transformation field f is parallel (Walker 1955, 1958) $D_X f = 0$, then $\alpha(Y)\,\alpha(HX) = h(X, Y)$ from which h is symmetric and $\alpha(Y)\,\alpha(HX) = \alpha(X)\,\alpha(HY)$. So $\alpha(X)\,h(U, U) = \alpha(HX)$. Setting $\mu = h(U, U)$, we get $h(X, Y) = \mu\alpha(X)\,\alpha(Y)$. From which $h(X, U) = \mu\alpha(X)$. Thus putting $Y = U$ in $\alpha(Y)\; HX - h(X, Y)\; U = 0$, we find that $HX = \mu\alpha(X)U$. On the other hand $w(x) = -\,h(X, V) = -\,\mu\alpha(X)\,\alpha(V) = 0$.

Let D' be the induced connexion on $b(P)$ with respect to the fundamental vector field E of the almost paracontact ambient space $M(\phi, E, \eta)$. Then the equations of Gauss and Weingarten are (Goldberg and Yano 1970)

$$(D'_X\, B)\,(Y) = h'(X, Y)\,E \qquad \text{...(3.6)}$$

$$D'_X\, E = -\, BH'X + w'(X)\,E. \qquad \text{...(3.7)}$$

From

$$(D_X B)\,(Y) = D_{BX} BY - BD_X Y,$$

$$(D'_X\, B)\,(Y) = D'_{BX}\, BY - B\,D'_X\, Y,$$

and

$$(D_X B)\,(Y) = h(X, Y)\,N,$$

$$(D'_X\, B)\,(Y) = h'(X, Y)\,(BV + \lambda N),$$

we have

$$B(D'_X\, Y - D_X Y) = -\, h'(X, Y)\, BV - [\lambda h'(X, Y) - h(X, Y)]\,N.$$

Therefore

$$D' = D - h' \otimes V, \;\; h = \lambda h'. \qquad \text{...(3.8)}$$

On the other hand from (3.7), we have

$$H' = \lambda H - DV + w' \otimes V$$

and

$$\lambda w' = h'(\cdot, V) + d\lambda + \lambda w.$$

Thus the hypersurface is totally flat with respect to the affine normals E and N if and only if

$$DV = w' \otimes V.$$

Theorem 3.6 — Let $P(J, \alpha)$ be a non-invariant hypersurface with para f-structure of an almost paracontact manifold with (ϕ, E, η) connexion. Then, if P is totally geodesic, J is parallel with respect to D.

PROOF : By (3.8), $h' = h = 0$ and $D' = D$. But $D'J = 0$ in a non-invariant hypersurface of an almost paracontact manifold with (ϕ, E, η)-connexion. So J is parallel with respect to D.

Theorem 3.7 — Let P be a non-invariant hypersurface with para (f, U, V, u, v, λ)-structure in an almost paracontact manifold satisfying (ϕ, E, η)-connexion. Then with respect to the induced connexion D on P, the quartic structure on P satisfies the relations

$$\left.\begin{aligned} &(D_X f) = - u(Y)\, HX - h(X, Y)\, U \\ &D_X V = \lambda HX,\ D_X U = fHX + w(X)\, U \\ &(D_X v)(Y) = \lambda h(X, Y) \\ &(D_X u)(Y) = h(X, fY) - w(X)\, u(Y) \\ &h(X, V) = - (X \cdot \lambda) - \lambda w(X) \\ &h(X, U) = - u(HX) \\ &v(HX) = \lambda w(X) - (X \cdot \lambda). \end{aligned}\right\} \qquad \ldots(3.9)$$

PROOF : By putting $J' = f$, $\eta(N) = \lambda$, $\alpha' = u$, $\beta = v$, differentiating $\eta(N) = \lambda$ along X and using Weingarten equation and $v(X) = \eta(BX)$, we get (3.9).

Theorem 3.8 — If the hypersurface P with (f, U, V, u, v, λ)-structure is affinely umbilical of an almost paracontact manifold with (ϕ, E, η)-connexion, it is totally flat if and only if $w = d(\log \lambda)$.

PROOF : Since P is affinely umbilical, $H = \mu I$. Now from $\lambda w(X) - (X \cdot \lambda) = v(HX) = \mu v(X)$, when $X \cdot \lambda - \lambda w(X) = 0$, $\mu = 0$ because $v(X) \neq 0$. Hence

$$w(X) = X \log \lambda = d(\log \lambda)\,(X)$$

or $$w = d(\log \lambda).$$

Conversely, if $w = d(\log \lambda)$

$$X \cdot \lambda - \lambda w(X) = 0 \Rightarrow \mu = 0 \ \text{ since } \ v(X) \neq 0.$$

Theorem 3.9 — Let P with para-(f, U, V, u, v, λ)-structure be a non-invariant hypersurface of an almost paracontact structure with (ϕ, E, η) connexion. Then if the linear transformation field f is parallel, we have

$$(1 - \lambda^2)^2\, h(X, Y) = \mu u(X)\, u(Y) \qquad \ldots(3.10a)$$

$$(1 - \lambda^2)^2\, HX = - \mu u(X)\, U \qquad \ldots(3.10b)$$

and

$$w = - d(\log \lambda) \qquad \ldots(3.10c)$$

PROOF : Since linear transformation field f is parallel from (3.9), we have

$$u(Y)\,u(HX) = -\,(1-\lambda^2)\,h(X, Y). \qquad ...(3.11)$$

Since h is symmetric

$$u(Y)\,u(HX) = u(X)\,u(HY),$$

so

$$-\,u(X)\,h(U, U) = (1-\lambda^2)\,u(HX), \qquad ...(3.12)$$

setting $\mu = h(U, U)$ and using (3.11), (3.12), we get (3.10a). From (3.10a) putting $Y = U$, we get

$$(1-\lambda^2)\,h(X, U) = \mu u(X). \qquad ...(3.13)$$

Again putting $Y = U$ in $u(Y)\,HX + h(X, Y)\,U = 0$ and using (3.13), we get (3.10b).

Putting $Y = V$ in $u(Y)\,HX + h(X, Y)\,U = 0$, we get

$$h(X, V) = 0.$$

Then we have

$$\lambda w(X) = -\,(X \cdot \lambda)$$

which gives

$$w = -\,d(\log \lambda).$$

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