AN APPLICATION OF THE THEOREM OF SRIVASTAVA AND BRENNER

by ARUNA SRIVASTAVA, *Department of Mathematics, University of Delhi, Delhi* 110007

(*Received* 29 *September* 1977; *after revision* 18 *February* 1978)

The upper and lower bounds for the determinant of a dominant diagonal matrix have been used recently by Srivastava and Brenner (1974) to obtain bounds on classical orthogonal polynomials. Similar methods are used here on the generalized Pasterneck polynomials.

Recently, Srivastava and Brenner (1974) have applied certain results due to Price (1951) and Brenner (1954) in order to obtain bounds for a general system of polynomials $p_n(x)$ satisfying a three-term recurrence relation of the type

$$p_{n+1}(x) = (A_n x + B_n)\, p_n(x) - C_n p_{n+1}(x). \qquad ...(1.1)$$

In the present note, using the results (12) and (14) of Srivastava and Brenner (1974, p. 374, Th. 1 and 2), we have obtained bounds for generalized Pasterneck's polynomials. For that, we first assume that the generalized Pasterneck polynomials defined by

$$\begin{aligned} P_n(x) &= p_n(x, \alpha, \beta, \beta_2\ ;\ 1) \\ &= {}_3F_2\,[\,-n, n+\alpha, x\ ;\ \beta_1, \beta_2\ ;\ 1] \\ &= \sum_{r=0}^{n} \frac{(-n)_r\,(n+\alpha)_r\,(x)_r}{r!\,(\beta_1)_r\,(\beta_2)_r} \end{aligned} \qquad ...(1.2)$$

satisfy (1.1), so that we have

$$\begin{aligned} &\sum_{r=0}^{n+1} \frac{(-n-1)_r\,(n+\alpha+1)_r\,(x)_r}{r!\,(\beta_1)_r\,(\beta_2)_r} \\ &\quad = (A_n x + B_n) \sum_{r=0}^{n} \frac{(-n)_r\,(n+\alpha)_r\,(x)_r}{r!\,(\beta_1)_r\,(\beta_2)_r} \\ &\qquad - C_n \sum_{r=0}^{n-1} \frac{(-n+1)_r\,(n+\alpha-1)_r\,(x)_r}{r!\,(\beta_1)_r\,(\beta_2)_r}. \end{aligned} \qquad ...(1.3)$$

Writing $x \cdot (x)_r = (x)_{r+1} - r(x)_r$ in (1.3) and equating the coefficients of $(x)_r$ (Jorden 1950, p. 169) we get

$$(n+1)(n+\alpha+r) = (n+\alpha)(n-r+1)\,B_n$$
$$- r(n+\alpha)\left[\frac{(\beta_1+r-1)(\beta_2+r-1)}{n+\alpha+r-1} + n - r + 1\right] A_n$$
$$+ \frac{(n+\alpha)(n-r+1)(n+\alpha-1)(n-r)}{n(n+\alpha+r-1)}\,C_n \qquad \ldots(1.4)$$

which, on putting $r = 0$, $n+1$ and n, gives

$$B_n + C_n = 1 \qquad \ldots(1.5)$$

$$A_n = -\frac{(2n+\alpha)(2n+\alpha+1)}{(n+\alpha)(\beta_1+n)(\beta_2+n)} \qquad \ldots(1.6)$$

and

$$B_n = (2n+\alpha)\,D_n/\{(n+\alpha)(2n+\alpha-1)(\beta_1+n)(\beta_2+n)\} \qquad \ldots(1.7)$$

where

$$D_n = (n+1)(2n+\alpha-1)(\beta_1+n)(\beta_2+n)$$
$$- n(2n+\alpha-1)(2n+\alpha+1)$$
$$- n(2n+\alpha+1)(\beta_1+n-1)(\beta_2+n-1). \qquad \ldots(1.8)$$

With A_n, B_n and C_n defined by (1.5), (1.6) and (1.7), the inequalities (12) and (14) of Srivastava and Brenner (1974, p. 374, Th. 1 and 2) are satisfied when $|x| \geqslant R + S_1$ and $|x| \geqslant R + S_2$, respectively, where

$$R = \max_{0 \leqslant j \leqslant n} \left| D_j/\{(2j+\alpha+1)(2j+\alpha-1)\} \right| \qquad \ldots(1.9)$$

$$S_1 = \max_{0 \leqslant j \leqslant n-1} \left| \frac{(j+\alpha)(j+\beta_1)(j+\beta_2)}{(2j+\alpha)(2j+\alpha+1)} \right|$$
$$+ \max_{1 \leqslant j \leqslant n} \left| \frac{(j+\alpha)(2j+\alpha-1)(j+\beta_1)(j+\beta_2) - (2j+\alpha)\,D_j}{(2j+\alpha-1)(2j+\alpha+1)(2j+\alpha)} \right| \qquad \ldots(1.10)$$

$$S_2 = \max_{1 \leqslant j \leqslant n} \left| \frac{(j+\alpha)(j+\beta_1)(j+\beta_2)}{(2j+\alpha)(2j+\alpha+1)} \times \left(\frac{(j+\alpha)(2j+\alpha-1)(\beta_1+j)(\beta_2+j) - (2j+\alpha)\,D_j}{(2j+\alpha-1)(2j+\alpha+1)(2j+\alpha)} \right) \right|^{1/2}$$
$$+ \max_{0 \leqslant j \leqslant n-1} \left| \frac{(j+\alpha)(\beta_1+j)(\beta_2+j)}{(2j+\alpha)(2j+\alpha+1)} \times \frac{(j+\alpha+1)(2j+\alpha+1)(\beta_1+j+1)(\beta_2+j+1) - (2j+\alpha+2)\,D_{j+1}}{(2j+\alpha+1)(j+\alpha+1)(\beta_1+j+1)(\beta_2+j+1)} \right|^{1/2}. \qquad \ldots(1.11)$$

The value of D_j can be obtained from (1.8).

Using (1.9), (1.10) and (1.11), we have the following consequences of Theorems 1 and 2 of Srivastava and Brenner (1974, p. 374).

Theorem A — For $n \geqslant 1$ and $|x| \geqslant R + S_1$, the generalized Pasterneck polynomial satisfies the inequalities

$$(1)\quad \prod_{j=1}^{n} \{ |A_j x + B_j| - |C_j| \}$$
$$\leqslant |P_{n+1}(x)| \; |\beta_1\beta_2| \; |\beta_1\beta_2 - x(\alpha + 1)|^{-1}$$
$$\leqslant \prod_{j=1}^{n} \{ |A_j x + B_j| + |C_j| \}. \qquad \ldots(1.12)$$

Theorem B — For $n \geqslant 1$ and $|x| \geqslant R + S_2$, the generalized Pasterneck polynomials satisfies the inequalities

$$(2)\quad \prod_{j=1}^{n} \{ |A_j x + B_j| - |\sqrt{C_j}| \}$$
$$\leqslant |p_{n+1}(x)| \; |\beta_1\beta_2| \; |\beta_1\beta_2 - x(1 + \alpha)|^{-1}$$
$$\leqslant \prod_{j=1}^{n} \{ |A_j x + B_j| + |\sqrt{C_j}| \} \qquad \ldots(1.13)$$

A_j, B_j and C_j are given by (1.5), (1.6), (1.7).

The bounds from (1.13) are finer than those obtainable from (1.12).

If we put $x = \frac{1}{2}(l + z + m)$, $\alpha = \beta_1$, $\beta_2 = m + 1$ in (1.2), $p_n(x)$ reduces to Pasterneck's polynomial (Rainville 1967, p. 291); further, the substitution $m = 0$ reduces it to Bateman's polynomial (Rainville 1967, p. 289). Corresponding substitutions in (1.12) and (1.13) will yield the inequalities for these polynomials.

Acknowledgement

The author is highly grateful to the referee for worthy suggestions.

References

Brenner, J. L. (1954). A bound for a determinant with dominant main diagonal. *Proc. Am. math. Soc.*, **5**, 631-34.

Jorden, Charles (1950). Calculus of Finite Differences. Cheslea Publishing Co., New York.

Price, G. B. (1951). Bounds for determinants with dominant principal diagonal. *Proc. Am. math. Soc.*, **2**, 497-502.

Rainville, E. D. (1967). Special Functions. Macmillan & Co., N.Y.

Srivastava, H. M., and Brenner, J. L. (1974). Bounds for Jacobi and related polynomials derivable by matrix methods. *J. Approximation Theory*, **12**, 372-77.