EXPANSION OF KAMPÉ DE FÉRIET FUNCTION IN TERMS OF GENERALIZED PROLATE SPHEROIDAL WAVE FUNCTION AND ITS APPLICATIONS*

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(*Received* 29 *November* 1977)

The object of the present paper is to establish an expansion formula for Kampé de Fériet function in terms of generalized prolate spheroidal wave function (Gupta 1977). For deriving this expansion formula we have first evaluated a desired integral. This expansion formula being of very general nature can be transformed to yield many new results involving various commonly used functions, viz spheroidal wave functions, Jacobi, Legendre polynomials, etc., occurring in Applied Mathematics and Mathematical Physics.

Finally, as an application of our results, we have obtained expansions of Coulumb wave functions (Curtiz 1964) for all values of angular momentum quantum number (L) as a product of generalized confluent hypergeometric function and the generalized prolate spheroidal wave function.

1. INTRODUCTION AND NOTATIONS

The generalized prolate spheroidal wave function has been defined by Gupta (1977) as the solution of the differential equation :

$$(1 - x^2) \frac{d^2y}{dx^2} + [\beta - \alpha - (\alpha + \beta + 2) x] \frac{dy}{dx} + [\chi_n^{(\alpha,\beta)}(c) - c^2x^2] y = 0 \qquad \ldots(1.1)$$

in the form of an infinite sum :

$$Y(x) = \phi_n^{(\alpha,\beta)}(c, x) = p_n^{(\alpha,\beta)}(x) + \sum_{j=1}^{\infty} c^{2j} Q_j(\alpha, \beta, n, x). \qquad \ldots(1.2)$$

The eigenvalue and eigenfunction for eqn. (1.1) have been given by

$$\chi_n^{(\alpha,\beta)}(c) = n(n + \alpha + \beta + 1) + \sum_{j=1}^{\infty} c^{2j} a_j(\alpha, \beta, n) \qquad \ldots(1.3)$$

where Q_j's and a_j's are independent of c.

As shown in appendix of Slepian (1964), Q's and a's can be determined recursively in an elementary manner.

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*Presented at the 65th Indian Science Congress, Ahmedabad.

The series (1.2) can be regrouped to give

$$\phi_n^{(\alpha,\beta)}(c, x) = \sum_{j=0}^{\infty} d_{j,n}^{(\alpha,\beta)}(c)\, p_{n+j}^{(\alpha,\beta)}(x). \qquad \text{...(1.4)}$$

Substituting (1.4) in (1.1) we get a five term recursion formula for the coefficients $d_{j,n}^{(\alpha,\beta)}$ in a manner quite parallel to the case of differential equation satisfied by so-called prolate spheroidal wave functions. The method of Bouwkamp (1957) can be adopted and used in a fruitful way for the computation of $d_{j,n}^{(\alpha,\beta)}(c)$ and the eigen-values $\chi_n^{(\alpha,\beta)}(c)$.

The orthogonality property of generalized prolate spheroidal wave functions (Gupta 1977) is given by

$$\int_0^1 x^{\alpha}(1-x)^{\beta}\, \phi_m^{(\alpha,\beta)}(c, 1-2x)\, \phi_n^{(\alpha,\beta)}(c, 1-2x)\, dx$$

$$= \begin{cases} 0, & m \neq n \\ N_n^{(\alpha,\beta)}, & m = n \end{cases} \qquad \text{...(1.5)}$$

where

$$N_n^{(\alpha,\beta)} = \sum_{j=0}^{\infty} [d_{j,n}^{(\alpha,\beta)}(c)]^2$$

$$\times \frac{\Gamma(1+\alpha+n+j)\,\Gamma(1+\beta+n+j)}{(n+j)!\,(\alpha+\beta+2n+2j+1)\,\Gamma(\alpha+\beta+n+j+1)}. \qquad \text{...(1.6)}$$

The Coulomb Wave Functions — The so-called Coulomb wave functions are essentially confluent hypergeometric functions. The regular and irregular forms of these functions may be expressed in terms of Whittaker functions by the equations (Abramowitz and Stegun 1964, Curtiz 1964, Morse and Feshbach 1953)

$$F_L(\eta, z) = C_L(\eta)\, z^{1+L}\, e^{-iz}\, {}_1F_1\left(\begin{matrix} 1+L+i\eta\ ; \\ 2L+2\ \ ; \end{matrix}\ 2iz\right) \qquad \text{...(1.7a)}$$

$$= C_L(\eta)\,(2i)^{-L-1}\, M_{i\eta,\, L+(1/2)}(2iz) \qquad \text{...(1.7b)}$$

$$G_L(\eta, z) = \frac{2^L}{C_L(\eta)\,(2L+1)!}\, \mathrm{Re}\,[\Gamma(L+1-i\eta)\, i^L\, W_{i\eta, L+(1/2)}(2iz)] \qquad \text{...(1.8a)}$$

(*equation continued on p.* 766)

$$= \frac{2^L}{C_L(\eta)\,(2L+1)!}\,\mathrm{Re}\Big[\Gamma(1+L-i\eta)\,i^L\Big\{\frac{\Gamma(-1-2L)}{\Gamma(-L-i\eta)}$$

$$\times M_{i\eta,L+(1/2)}(2iz)+\frac{\Gamma(1+2L)}{\Gamma(1+L-i\eta)}\,M_{i\eta,-L-(1/2)}(2iz)\Big\}\Big] \quad \ldots(1.8b)$$

$$C_L(\eta)=\frac{2^L\,e^{-\pi\eta/2}\,|\,\Gamma(L+1+i\eta)\,|}{(2L+1)!} \quad \ldots(1.9)$$

where $\eta = ZZ'/hv$; $z = uv/h$; v being the relative velocity of the colliding particles, r the distance between them, u the reduced mass and Ze, $Z'e$ their charges. In the applications, L is usually a positive integer or zero denoting the orbital angular momentum quantum number.

For further details on Coulomb wave functions, *see* Abramowitz and Stegun (1964) and Curtiz (1964). We have followed the notations of the former.

We shall make use of the familiar abbreviation

$$(a)_m = \Gamma(a+m)/\Gamma(a) = a(a+1)\ldots(a+m-1)$$

and in what follows for the sake of brevity we shall express the Kampé de Fériet function in the notation of Burchnall and Chaundy (1941).

$$F\left(\begin{matrix}(a),(a'):(c)\,;\,(c')\,; \\ \\ (b),(b'):(d)\,;\,(d')\,;\end{matrix}\quad x,y\right)$$

$$=\sum_{m,n=0}^{\infty}\frac{((a))_{m+n}\,((a'))_{m+n}\,((c))_m\,((c'))_n\,x^m\,y^n}{((b))_{m+n}\,((b'))_{m+n}\,((d))_m\,((d'))_n\,m!\,n!}$$

where

$$A + A' + C \leqslant B + B' + D$$

and

$$C' + A + A' \leqslant B + B' + D'.$$

Also, (a) is taken to denote the sequence of A parameters $a_1, \ldots, a_A$, i.e. unless stated otherwise there are A of a parameters, A' of a' parameters and so on. Thus $((a))_m$ is to be interpreted as $\prod_{j=1}^{A} (a_j)_m$, with similar interpretations for $((a'))_m$, etc.

2. Integral Required

We evaluate here the following integral required for establishing the desired expansion :

$$I_n(\rho) = \int_0^1 z^{\rho}(1-z)^{\beta}\, \phi_n^{(\alpha,\beta)}(c', 1-2z) \times F\left(\begin{matrix}(a),(a'):(c);(c');\\ \\ (b),(b'):(d);(d');\end{matrix}\; uz, vz\right) dz$$

$$= \frac{\Gamma(1+\rho)}{\Gamma(\alpha-\rho)} \sum_{j=0}^{\infty} d_{j,n}^{(\alpha,\beta)}(c') \frac{\Gamma(1+\beta+n+j)\,\Gamma(\alpha-\rho+n+j)}{(n+j)!\,\Gamma(\beta+\rho+n+j+2)}$$

$$\times F\left(\begin{matrix}1+\rho, 1+\rho-\alpha, (a), (a'):(c);(c');\\ \\ 1+\rho-\alpha-n-j, \beta+\rho+n+j+2, (b),(b'):(d);(d');\end{matrix}\; u, v\right) \quad ...(2.1)$$

with conditions of validity as :

$$A + A' + C \leqslant B + B' + D,\ A + A' + C' \leqslant B + B' + D,$$
$$\mathrm{Re}\,(\rho) > -1,\ \mathrm{Re}\,(\beta) > -1,\ \mathrm{Re}\,(\alpha) > -1,$$

and the series on the right-hand side of (2.1) is convergent.

PROOF : To evaluate the integral (2.1), we first express the generalized prolate spheroidal wave function and the Kampé de Fériet function in their series form (1.4) and (1.10), respectively, and change the order of integration and summations which is justified due to uniform convergence of the series involved in the process (Carslaw 1930). Then, evaluating the inner integral with the help of (Erdelyi *et al.* 1954, p. 284) we get the desired result.

3. DESIRED EXPANSION

Here we establish the following expansion formula :

$$z^{\rho}\, F\left(\begin{matrix}(a),(a'):(c);(c');\\ \\ (b),(b'):(d);(d');\end{matrix}\; uz, vz\right)$$

$$= \sum_{r=0}^{\infty} \frac{I_r(\alpha+\rho)}{N_r^{(\alpha,\beta)}}\, \phi_r^{(\alpha,\beta)}(c', 1-2z) \quad ...(3.1)$$

where, for convenience, $I_r(\alpha+\rho)$ denotes the value of the integral (2.1) when, ρ is replaced by $(\alpha+\rho)$ and $N_r^{(\alpha,\beta)}$ is defined by (1.6). This expansion formula is valid, when $A + A' + C \leqslant B + B' + D$, $A + A' + C' \leqslant B + B' + D'$, $\mathrm{Re}\,(z+\alpha) > -1$,

$\text{Re}(\alpha) > -1$, $\text{Re}(\beta) > -1$, $\rho \geqslant 0$, and the series on the right-hand side of (3.1) is convergent.

PROOF : To prove (3.1), let

$$f(z) = z^{\rho} F\left(\begin{matrix} (a), (a') : (c) ; (c') \; ; \\ \\ (b), (b') : (d) ; (d') \; ; \end{matrix} \; uz, vz\right)$$

$$= \sum_{r=0}^{\infty} M_r \; \phi_r^{(\alpha,\beta)} (c', 1 - 2z). \qquad \text{...(3.2)}$$

Equation (3.2) is valid since $f(z)$ is continuous and of bounded variation in the open interval (0, 1), where $\rho \geqslant 0$.

Multiplying both sides of (3.2) by $z^{\alpha}(1 - z)^{\beta} \phi_n^{(\alpha,\beta)} (c', 1 - 2z)$ and integrating between the limits 0 to 1, we get

$$\int_0^1 z^{\rho+\alpha}(1 - z)^{\beta} \phi_n^{(\alpha,\beta)} (c', 1 - 2z)$$

$$\times F\left(\begin{matrix} (a), (a') : (c) ; (c') \; ; \\ \\ (b), (b') : (d) ; (d') \; ; \end{matrix} \; uz, vz\right) dz$$

$$= \sum_{r=0}^{\infty} M_r \int_0^1 z^{\alpha}(1 - z)^{\beta} \phi_n^{(\alpha,\beta)} (c', 1 - 2z)$$

$$\times \phi_r^{(\alpha,\beta)} (c', 1 - 2z) \, dz. \qquad \text{...(3.3)}$$

Now, using the result (2.1) on the left and the orthogonality property (1.5) on the right-hand side of (3.3), we get

$$M_n = \frac{I_n(\rho + \alpha)}{N_n^{(\alpha,\beta)}} \qquad \text{...(3.4)}$$

where $I_n(\rho + \alpha)$ and $N_n^{(\alpha,\beta)}$ are defined by (2.1) and (1.5), respectively.

Hence, the expansion formula (3.1) follows immediately from (3.2) and (3.4).

4. APPLICATIONS

1. For the study of nuclear reactions involving positively charged particles like protons, deuterons, α-particles and heavy ions, the knowledge of Coulomb wave

functions is of great importance. So, as an application of our expansion formula (3.1), we evaluate here expansions for both regular and irregular wave functions as a product of generalized prolate spheroidal wave functions and generalized confluent hypergeometric functions for all values of L.

In eqn. (3.1), if we set $A = A' = B = B' = C' = D' = 0$, $C = D = 1$, $c_1 = 1 + L - i\eta$, $d_1 = 2L + 2$, $u = 2i$, $v = 0$, we get

$$z^{\rho}\,{}_1F_1\left(\begin{matrix}1 + L - i\eta; \\ \\ 2L + 2 \quad ;\end{matrix}\; 2iz\right) = \sum_{r=0}^{\infty} \frac{I'_r(\alpha + \rho)}{N_r^{(\alpha,\beta)}} \phi_r^{(\alpha,\beta)}(c', 1 - 2z) \quad \ldots(4.1)$$

where

$$I'_r(\alpha + \rho) = \frac{\Gamma(1 + \rho)}{\Gamma(\alpha - \rho)} \sum_{j=0}^{\infty} d_{j,n}^{(\alpha,\beta)}(c') \frac{\Gamma(1 + \beta + n + j)\,\Gamma(\alpha - \rho + n + j)}{(n + j)!\,\Gamma(\beta + \rho + n + j + 2)}$$

$$\times\; {}_3F_3\left(\begin{matrix}1 + \rho,\; 1 + \rho - \alpha,\; 1 + L - i\eta; \\ \\ 1 + \rho - \alpha - n - j,\; \beta + \rho + n + j + 2,\; 2L + 2;\end{matrix}\; 2i\right). \quad \ldots(4.2)$$

Now with the help of eqn. (1.7a), we finally get the following desired expansion :

$$F_L(\eta, z) = e^{-i\rho}\, C_L(\eta)\, z^{1+L-\rho} \sum_{r=0}^{\infty} \frac{I'_r(\alpha + \rho)}{N_r^{(\alpha,\beta)}} \phi_r^{(\alpha,\beta)}(c', 1 - 2z) \quad \ldots(4.3)$$

provided that $\text{Re}\,(\alpha + \rho) > -1$, $\text{Re}\,(\beta) > -1$, $\text{Re}\,(\alpha) > -1$, and $\rho \geqslant 0$.

To evaluate the corresponding expansion formula for irregular Coulomb wave function, we rewrite eqn. (1.8*b*) by virtue of (1.7*b*) in the following form :

$$G_L(\eta, z) = \frac{2^L}{(2L + 1)!} \text{Re}\Big[\Gamma(1 + L - i\eta)\, i^L \Big\{\frac{\Gamma(-1 - 2L)\,(2i)^{L+1}}{\Gamma(-L - i\eta)\,[C_L(\eta)]^2}$$

$$\times F_L(\eta, z) + \frac{\Gamma(1 + 2L)\,(2i)^{-L}}{(1 + L - i\eta)\,C_L(\eta)\,C_{-L-1}(\eta)} F_{-L-1}(\eta, z)\Big\}\Big]. \quad \ldots(4.4)$$

By virtue of (4.3), the expansion formula for $G_L(\eta, z)$ immediately follows which is valid under the same conditions as specified for eqn. (3.1).

2. Finally, it would seem worthwhile to mention that the expansions for Coulomb wave functions in terms of several interesting special functions and

polynomials can be easily deduced from our formulas (4.3) and (4.4) by specializing the parameters of $\phi_n^{(\alpha,\beta)}(c', 1-2z)$ as follows :

(i) Ultraspherical polynomials, if $\alpha = \beta \neq 0$, $c' = 0$;

(ii) Jacobi polynomials, if $c' = 0$;

(iii) Tchebichef polynomials (of first and second kind), if $\alpha = \beta = \pm \frac{1}{2}$, $c' = 0$;

(iv) Chebeyshive polynomials, if $\beta = -\frac{1}{2}$, $\alpha = 0$, $c' = 0$;

(v) Legendre function, if $\beta = \alpha = 0$, $c' = 0$;

(vi) Associated Legendre function, if $\beta = \alpha = m$, $c' = 0$;

(vii) Spheroidal wave functions of zero order (Bouwkamp 1957), if $\beta = \alpha = 0$, $c' = 0$; etc., etc.

Remarks : The expansions (4.3) and (4.4) may be used for the computation of F_L and G_L for both small and large values of η, i.e. for both high and low energies. The results evaluated by us find applications not only in the derivation of Coulomb functions but are also useful in obtaining many new results involving various functions of great interest. We omit details.

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