BIFURCATION THEORY FOR TWO POPULATION MODELS

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In the present paper, a set of conditions for a pair of eigenvalues of a stability problem matrix to be purely imaginary are obtained. These conditions are applied to discuss the existence of bifurcation points for (i) time-delay prey-predator model and (ii) growth of a single population under the effect of pollution.

1. Determination of Points of Bifurcation in General Stability Theory

Consider a physical system governed by the system of differential equations

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n, c_1, c_2, \dots, c_m) \qquad (i = 1, 2, \dots, n) \qquad \dots(1)$$

where $c_1, c_2, \dots, c_m$ are certain parameters.

Its equilibrium points are given by

$$f_i(x_1, x_2, \dots, x_n, c_1, c_2, \dots, c_m) = 0 \qquad (i = 1, 2, \dots, n). \qquad \dots(2)$$

Let $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ be an equilibrium point.

Let $\quad x_i = \hat{x}_i + u_i \qquad (i = 1, 2, \dots, n). \qquad \dots(3)$

Substituting in (1), using (2) and neglecting squares, products and higher power of u_i's, we get

$$\frac{du_i}{dt} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} u_j = \sum_{j=1}^{n} a_{ij}u_j \text{ (say)} \qquad (i = 1, 2, \dots, n). \qquad \dots(4)$$

Substituting

$$u_i = A_i e^{\lambda t} \qquad (i = 1, 2, \dots, n) \qquad \dots(5)$$

we find that (5) will give a solution of (4) if

$$| A - \lambda I | = 0, \qquad \dots(6)$$

where A is the matrix

$$\begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \dots & \dots \dots \dots \\ \dots & \dots \dots \dots \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix}. \qquad \dots(7)$$

Let eqn. (6) be written as

$$\lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} \dots + K_n = 0 \qquad \dots(8)$$

where $K_1, K_2, \dots, K_n$ are functions of the parameters $c_1, c_2, \dots, c_m$. For discussing bifurcation, we are interested in finding whether (7) or (8) can have purely imaginary roots of the form $\pm i\omega$. The interest arises because the nature of the equilibrium point changes when parameter values change slightly in this case. For the two-dimensional case if the roots are $\nu \pm i\omega$, we get an unstable focus if $\nu > 0$, a centre if $\nu = 0$ and a stable node if $\nu < 0$. ν is a function of the parameters and for some values of the parameters, ν can be zero. A slight change in the values of the parameters would change the nature of the equilibrium.

If $\pm\omega i$ are roots of (8), we get

$$(\omega^n - K_2\omega^{n-2} + K_4\omega^{n-4} \dots) \pm i(K_1\omega^{n-1} - K_3\omega^{n-3} + \dots) = 0. \qquad \dots(9)$$

Equating real and imaginary parts seperately to zero, we get

$$\left.\begin{aligned} &\omega^n - K_2\omega^{n-2} + K_4\omega^{n-4} \dots + (-1)^{(n-1)/2} K_{n-1}\,\omega = 0 \\ &K_1\omega^{n-1} - K_3\omega^{n-3} + K_5\omega^{n-5} \dots + (-1)^{(n-1)/2} K_n = 0, \end{aligned}\right\} \qquad \dots(10)$$

if n is odd and

$$\left.\begin{aligned} &\omega^n - K_2\omega^{n-2} + K_4\omega^{n-4} \dots (-1)^{n/2} K_n = 0 \\ &K_1\omega^{n-1} - K_3\omega^{n-3} + K_5\omega^{n-5} \dots (-1)^{(n-2)/2} K_{n-1}\omega = 0, \end{aligned}\right\} \qquad \dots(11)$$

if n is even.

For (8) to have a pair of purely imaginary roots, eqns. (10) or (11) should have a common real solution.

2. Special Cases

For $n = 2$

$$\omega^2 - K_2 = 0, \; K_1\omega = 0. \qquad \dots(12)$$

The required conditions are

$$K_1 = 0,\ K_2 > 0. \qquad \text{...(13)}$$

For $n = 3$

$$\omega^3 - K_2\omega = 0,\ K_1\omega^2 - K_3 = 0. \qquad \text{...(14)}$$

The required conditions are

$$K_2 > 0, \frac{K_3}{K_1} > 0, K_2 = \frac{K_3}{K_1}. \qquad \text{...(15)}$$

For $n = 4$

$$\omega^4 - K_2\omega^2 + K_4 = 0,\ K_1\omega^3 - K_3\omega = 0. \qquad \text{...(16)}$$

The required conditions are

$$\frac{K_3}{K_1} > 0,\ K_3^2 - K_1K_2K_3 + K_4K_1^2 = 0. \qquad \text{...(17)}$$

For $n = 5$

$$\omega^5 - K_2\omega^3 + K_4\omega = 0,\ K_1\omega^4 - K_3\omega^2 + K_5 = 0 \qquad \text{...(18)}$$

or

$$\frac{\omega^4}{-K_2K_5 + K_3K_4} = \frac{\omega^2}{K_4K_1 - K_5} = \frac{1}{-K_3 + K_1K_2}.$$

The required conditions are

$$\frac{K_4K_1 - K_5}{K_1K_2 - K_3} > 0 \qquad \text{...(19a)}$$

$$(K_1K_4 - K_5)^2 = (K_3K_4 - K_2K_5)(K_1K_2 - K_3). \qquad \text{...(19b)}$$

For $n = 6$

$$\omega^6 - K_2\omega^4 + K_4\omega^2 - K_6 = 0,\ K_1\omega^5 - K_3\omega^3 + K_5\omega = 0. \qquad \text{...(20)}$$

These give

$$K_3\omega^4 - \omega^2(K_2K_3 + K_5 - K_1K_4) + K_2K_5 - K_1K_6 = 0$$

and

$$K_1\omega^4 - \omega^2K_3 + K_5 = 0$$

or

$$\frac{\omega^4}{K_3(K_2K_5 - K_1K_6) - K_5(K_2K_3 + K_5 - K_1K_4)}$$

$$= \frac{\omega^2}{K_1(K_2K_5 - K_1K_6) - K_3K_5} = \frac{1}{-K_3^2 + K_1(K_2K_3 + K_5 - K_1K_4)}.$$

The required conditions are

$$\frac{K_1K_2K_5 - K_1^2K_6 - K_3K_5}{K_1K_2K_3 + K_1K_5 - K_1^2K_4 - K_3^2} > 0 \quad \text{...(21a)}$$

and

$$(K_1K_2K_5 - K_1^2 K_6 - K_3K_5)^2 = (K_1K_4K_5 - K_3K_1K_6 - K_5^2) \times (- K_3^2 + K_1K_2K_3 + K_1K_5 - K_1^2K_4). \quad \text{...(21b)}$$

For $n = 7$

$$\omega^7 - K_2\omega^5 + K_4\omega^3 - K_6\omega = 0 \quad \text{...(22a)}$$

$$K_1\omega^6 - K_3\omega^4 + K_5\omega^2 - K_7 = 0. \quad \text{...(22b)}$$

These give

$$(- K_3 + K_1K_2)\, \omega^4 + (K_5 - K_1K_4)\, \omega^2 - (K_7 - K_6K_1) = 0 \quad \text{...(23a)}$$

and

$$\omega^4 [K_5 - K_1K_4 + K_1K_2^2 - K_2K_3] - \omega^2 [K_7 - K_6K_1 + K_1K_2K_4 - K_3K_4] + (K_6K_1K_2 - K_6K_3) = 0. \quad \text{...(23b)}$$

The required conditions are

$$\frac{K_6K_1K_5 - K_1^2 K_6K_4 + K_1K_4K_7 - K_5K_7 - K_1K_2^2 K_7 + K_2K_3K_7 + K_1K_2K_3K_6 - K_3^2K_6}{- K_3K_6K_1 + K_3K_7 - K_3^2 K_4 + K_1K_2K_3K_4 + K_1^2 K_2K_6 - K_1K_2K_7 - K_5^2 - K_1K_2^2 K_5 + K_2K_3K_5 - K_1^2 K_4^2 + 2K_1K_4K_5} > 0 \quad \text{...(24a)}$$

and

$$(K_6K_1K_5 - K_1^2 K_6K_4 - K_5K_7 - K_1K_2^2 K_7 + K_2K_3K_7 + K_1K_2K_3K_6 - K_3^2 K_6 + K_1K_4K_7)^2$$
$$= (K_1K_2K_6K_5 - K_3K_6K_5 - K_6^2 K_1^2 + 2K_1K_6K_7 - K_7^2 + K_3K_4K_7 - K_1K_2K_4K_7)$$
$$(- K_3K_6K_1 + K_3K_7 - K_3^2 K_4 + K_1K_2K_3K_4 + K_1^2 K_2K_6 - K_1K_2K_7 - K_5^2 + 2K_1K_4K_5 - K_1K_2^2 K_5 + K_2K_3K_5 - K_1^2 K_4^2). \quad \text{...(24b)}$$

Thus we find that in every case we get two conditions. One of which is an inequality and other is an equation. The second condition involves $c_1, c_2, \ldots, c_m$. If $(m - 1)$ of the parameters are kept fixed, this will determine the value of the mth parameter. If the value of this parameter is changed slightly from this value, the nature of the equilibrium will change, although the first condition will continue to be satisfied.

3. Time-delay Predator-prey Models

Time-delay predator-prey models have been discussed by Wangersky and Cunningham (1957) and Macdonald (1976, 1977). In particular.

Macdonald (1977) has obtained the following system of equations for a time-delay predator-prey model

$$\frac{dN}{dt} = \epsilon N\left(1 - \frac{N}{K}\right) - \alpha NP \qquad \ldots(25a)$$

$$\frac{dP}{dt} = -\gamma P + \beta P Q_1 \qquad \ldots(25b)$$

$$\frac{dQ_i}{dt} = a(Q_{i-1} - Q_i) \qquad (i = 1, 2, \ldots, r-1) \qquad \ldots(25c)$$

$$\frac{dQ_r}{dt} = a(N - Q_r). \qquad \ldots(25d)$$

Making the substitutions

$$N = Kn,\ P = Kp,\ Q = Kq,\ \tau = \epsilon t, \qquad \ldots(26)$$

we get

$$\dot{n} = n(1-n) - bnp \qquad \ldots(27a)$$

$$\dot{p} = -cp + dpq_1 \qquad \ldots(27b)$$

$$\dot{q}_i = f(q_{i-1} - q_i) \qquad (i = 1, 2, \ldots, r-1) \qquad \ldots(27c)$$

$$\dot{q}_r = f(n - q_r) \qquad \ldots(27d)$$

where

$$b = \frac{K\alpha}{\epsilon},\ c = \frac{r}{\epsilon},\ d = \frac{K\beta}{\epsilon},\ f = \frac{a}{\epsilon}. \qquad \ldots(28)$$

The matrix A for this system is given by

$$\begin{bmatrix} -n^* & -bn^* & 0 & 0 & 0 \ldots 0 & 0 \\ 0 & 0 & dp^* & 0 & 0 \ldots 0 & 0 \\ 0 & 0 & -f & f & 0 \ldots 0 & 0 \\ 0 & 0 & 0 & -f & f \ldots 0 & 0 \\ 0 & 0 & 0 & 0 & -f \ldots 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \ \ldots\ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 \ldots -f & f \\ f & 0 & 0 & 0 & 0 \ldots 0 & -f \end{bmatrix}. \qquad \ldots(29)$$

The equation for eigenvalues on simplification, gives

$$\lambda^{r+2} + \lambda^{r+1}({}^rc_1 f + n^*) + \lambda^r({}^rc_2 f^2 + n^*\, {}^rc_1 f) + \lambda^{r-1}({}^rc_3 f^3 + n^*\, {}^rc_2 f^2) + \ldots + \lambda^2(f^r + n^*\, {}^rc_1 f^{n-1}) + \lambda n^* f^r + \alpha f^r = 0 \qquad \ldots(30)$$

where

$$n^* = q_1^* = q_2^* = \ldots = q_r^* = \frac{c}{d},\; p^* = \frac{d-c}{db},\; \alpha = \frac{c}{d}(d-c). \qquad \ldots(31)$$

4. Special Cases

(i) $r = 1$

$$\lambda^3 + \lambda^2(f + n^*) + \lambda n^* f + \alpha f = 0 \qquad \ldots(32)$$

(ii) $r = 2$

$$\lambda^4 + \lambda^3(2f + n^*) + \lambda^2(f^2 + 2n^* f) + \lambda n^* f^2 + \alpha f^2 = 0 \qquad \ldots(33)$$

(iii) $r = 3$

$$\lambda^5 + \lambda^4(3f + n^*) + \lambda^3(3f^2 + 3n^* f) + \lambda^2(f^3 + 3n^* f^2) + \lambda n^* f^3 + \alpha f^3 = 0 \qquad \ldots(34)$$

(iv) $r = 4$

$$\lambda^6 + \lambda^5(4f + n^*) + \lambda^4(6f^2 + 4n^* f) + \lambda^3(4f^3 + 6n^* f^2) + \lambda^2(f^3 + 4n^* f) + \lambda n^* f^6 + 2f^6 = 0. \qquad \ldots(35)$$

For discussing the existence of points of bifurcation, we apply the conditions (15), (17), (19) and (21).

For $r = 1$ the required condition is

$$f = \frac{d^2 - dc - c}{d}. \qquad \ldots(36)$$

Thus for a bifurcation point to exist, we should have

$$d^2 > (d + 1)\, c. \qquad \ldots(37)$$

For $r = 2$, the required condition is

$$-\, 2n^* f^3 + f^2(4\alpha - 4n^{*2}) + f(4n^*\alpha - 2n^{*3}) + \alpha n^{*2} = 0. \qquad \ldots(38)$$

This equation does have a positive real root and as such in this case, a bifurcation point does exist.

For $r = 3$, the required conditions are

$$\frac{n^* f^3(3f + n^*) - \alpha f^3}{(3f + n^*)(3f^2 + 3n^* f) - (f^3 + 3n^* f^2)} > 0 \qquad \ldots(39)$$

and

$$- 8n^*f^5 + f^4(24d - 33n^{*2}) + f^3(45n^*\alpha - 24n^{*3}) + f^2(34n^{*2}\alpha - 8n^{*4} + \alpha^2) + f(9n^{*3}\alpha) + 9n^{*2} = 0. \quad \ldots(40)$$

Equation (40) has at least one real positive root. If this satisfies eqn. (39), a point of bifurcation exists

For $r = 4$, the required conditions are

$$\frac{(4f + n^*)(6f^2 + 4n^*f)(n^*f^6) - (4f + n^*)^2\alpha f^6 - (4f^3 + 6n^*f^2)n^*f^6}{(4f + n^*)(6f^2 + 4n^*f)(4f^3 + 6n^*f^2) + (4f + n^*)n^*f^6 - (4f + n^*)^2(f^3 + 4n^*f) - (4f^3 + 6n^*f^2)^2} > 0 \quad \ldots(41)$$

and

$$\begin{aligned} &- 4n^{*3}f^{11} - f^{10}(+ n^{*4} + 480n^{*2}) + f^9(32n^{*2} + 576\alpha n^* - 824n^{*3}) \\ &+ f^8(16n^{*3} - 528n^{*4} + 704n^{*2}\alpha - 256\alpha^2 - 1280\alpha + 320n^*) \\ &+ f^7(816n^{*2} - 152n^{*5} + 128n^{*3} - 64n^* - 5184\alpha n^* + 372\alpha n^{*3} \\ &\qquad + 256\alpha + 2n^{*4} - 256n^*\alpha^2) + f^6\,[65n^{*4} \\ &+ 632n^{*3} - 7424\alpha n^{*2} + 1232n^{*2} + 576\alpha n^* + 90\alpha n^{*4} \\ &- 16n^{*6} - 96n^{*2}\alpha^2) + f^5(208n^{*4} + 2156n^{*3} + 8n^{*5} \\ &- 236\alpha n^{*2} + 1024\alpha n^* - 2752\alpha n^{*3} + 8n^{*5}\alpha - 16n^{*3}\alpha^2) \\ &+ f^4(24n^{*5} + 2527n^{*4} - 384n^{*3} + 2304\alpha n^{*2} \\ &- 1344\alpha n^{*4} - n^{*4}\alpha^2 + 76\alpha n^{*3}) \\ &+ f^3(832n^{*5} - 96n^{*4} - 1024n^{*3} - 144\alpha n^{*5} + 1344\alpha n^{*3} + 6\alpha n^{*4}] \\ &+ f^2(96n^{*6} - 768n^{*4} + 304\alpha n^{*4} - 8n^{*5}) \\ &+ f(24\alpha n^{*5} - 192n^{*5}) - 16n^{*6} = 0. \end{aligned} \quad \ldots(42)$$

Equation (42) will have no real positive root or it will have even number of real positive roots. If these satisfy eqn. (41), an even number of a points of bifurcation will exist.

5. Bifurcation Theory for the Growth of Population in the Presence of Pollution

Following May (1973), we take

$$\frac{dN}{dt} = kN - bN^2 - CN \int_{-\infty}^{t} N(t')\,G(t - t')\,dt \quad \ldots(43)$$

the kernal function $G(t, t')$ gives the influence of the population at time t' on the population at time t.

Let
$$G(t - t') = \frac{e^{-a(t-t')}\,(t - t')^{p-1}\,a^p}{\Gamma(p)} \qquad \ldots(44)$$

and

$$Q_p(t) = \int_{-\infty}^{t} N(t')\,\frac{e^{-a(t-t')}\,(t - t')^{p-1}\,a^p}{\Gamma(p)}\,dt \qquad \ldots(45)$$

then we have

$$\begin{aligned}
\frac{dN}{dt} &= kN(t) - bN^2(t) - CN(t)\,Q_p(t)\\
\frac{dQ_p}{dt} &= a(Q_{p-1}(t) - Q_p(t))\\
\frac{dQ_{p-1}}{dt} &= a(Q_{p-2}(t) - Q_{p-1}(t))\\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
\frac{dQ_2}{dt} &= a(Q_1(t) - Q_2(t))\\
\frac{dQ_1}{dt} &= a(N(t) - Q_1(t)).
\end{aligned} \qquad \ldots(46)$$

For equilibrium

$$\hat{N} = \frac{k}{b + c} = \hat{Q}_1 = \hat{Q}_2 \ldots = \hat{Q}_p \qquad \ldots(47)$$

The matrix A for the system is given by

$$\begin{bmatrix}
-b\hat{N} & -c\hat{N} & 0 & 0 & 0 \ldots 0 & 0\\
0 & -a & a & 0 & 0 \ldots 0 & 0\\
0 & 0 & -a & a & 0 \ldots 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 \quad -a & a\\
a & 0 & 0 & 0 & 0 \quad 0 & -a
\end{bmatrix} \qquad \ldots(48)$$

The equation for eigenvalues, on simplification, gives

$$\begin{aligned}
\lambda^{p+1} &+ \lambda^p({}^pc_1 a + b\hat{N}) + \lambda^{p-1}({}^pc_2 a^2 + {}^pc_1 ab\hat{N})\\
&+ \lambda^{p-2}({}^pc_3 a^3 + {}^pc_2 a^2 b\hat{N}) + \ldots\\
&+ b\hat{N}a^p + a^p c\hat{N} = 0.
\end{aligned} \qquad \ldots(49)$$

6. Special Cases

(i) $p = 1$

$$\lambda^2 + \lambda(a + b\hat{N}) + a\hat{N}\,(b + c) = 0. \quad \ldots(50)$$

(ii) $p = 2$

$$\lambda^3 + \lambda^2(2a + b\hat{N}) + \lambda(a^2 + 2ab\hat{N}) + a^2(b\hat{N} + c\hat{N}) = 0. \quad \ldots(51)$$

(iii) $p = 3$

$$\lambda^4 + \lambda^3(3a + b\hat{N}) + \lambda^2(3a^2 + 3ab\hat{N}) + \lambda(a^3 + 3a^2b\hat{N})$$
$$+ a^3(b\hat{N} + c\hat{N}) = 0. \quad \ldots(52)$$

(iv) $p = 4$

$$\lambda^5 + \lambda^4(4a + b\hat{N}) + \lambda^3(6a^2 + 4ab\hat{N}) + \lambda^2(4a^3 + 6a^2b\hat{N})$$
$$+ \lambda(a^4 + 4b\hat{N}a^3) + a^4\hat{N}\,(b + c). \quad \ldots(53)$$

For discussing the points of bifurcation, we apply conditions (13), (15), (17) and (19)

For $p = 1$

$$a + b\hat{N} = 0, \; a\hat{N}\,(b + c) > 0 \quad \ldots(54)$$

there is no real positive value of a so there will no point of bifurcation.

For $p = 2$

$$2a^2 + a(4b\hat{N} - c\hat{N}) + 2b^2\hat{N}^2 = 0 \quad \ldots(55)$$

This equation either has no positive real root or gives even number of real positive roots. If it gives even number of positive real roots, there exist two points of bifurcation.

For $p = 3$

$$\frac{a^3 + 3a^2b\hat{N}}{3a + b\hat{N}} > 0 \quad \ldots(56)$$

and

$$(a^3 + 3a^2b\hat{N})^2 - (3a + b\hat{N})\,(3a^2 + 3ab\hat{N})\,(a^3 + 3a^2b\hat{N})$$
$$+ a^3(b\hat{N} + c\hat{N})\,(3a + b\hat{N})^2 = 0 \quad \ldots(57)$$

$$- 8a^3 + a^2\hat{N}(9c - 24b) + a\hat{N}^2(6bc - 24b^2) + \hat{N}^3(b^2c - 8b^3) = 0. \quad \ldots(58)$$

If $c/b > 8$, point of bifurcation exist if that value of a also satisfies eqn. (56)

If $c/b < 8$, then either there will exist no positive real root or there exist even number of positive real roots. If roots exist and satisfy eqn. (56) there will exist a point of bifurcation.

For $p = 4$ the required conditions are

$$\frac{(a^4 + 4b\hat{N}a^3)(4a + b\hat{N}) - a^4\hat{N}(b + c)}{(4a + b\hat{N})(6a^2 + 4ab\hat{N}) - (4a^3 + 6a^2b\hat{N})} > 0 \qquad \text{...(59)}$$

and

$$64a^4 + \hat{N}a^3(256b - 112c) + a^2\hat{N}^2(384b^2 + 144bc - c^2) + a\hat{N}^3(256b^3 - 86b^2c) + \hat{N}^4(64b^4 - 16b^3c) = 0. \qquad \text{...(60)}$$

If $4 < c/b$ there exist a real positive root. If this satisfies eqn. (59) there exists a point of bifurcation. If $4 > c/b$ then either there will exist no positive real root or there exist even number of positive real roots. If roots exist and satisfy eqn. (59) there will exist a point of bifurcation.

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