ON A SPECIAL QUADRATIC STRUCTURE ON DIFFERENTIABLE MANIFOLDS

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From the quadratic structure a special quadratic structure has been defined in this paper; which is a generalization of an almost complex structure (Yano 1965). The existence and the integrability condition of this structure have been obtained.

Suppose M_n to be an n-dimensional differentiable manifold of differentiability class C^∞ and a linear transformation F [a tensor field of type (1, 1) of class C^∞] on the set $x(M)$ of vector fields on M_n given by the algebraic equation (Yano 1963)

$$F^2 + \alpha F + I = 0 \, (\alpha \in R) \qquad \ldots(1)$$

For an arbitrary vector-field $X \in x(M)$, the structural eqn. (1) can be expressed by

$$\bar{\bar{X}} + \alpha \bar{X} + X = 0, \qquad \ldots(1a)$$

where we define $\bar{X}$ as $F(X)$. For the sake of convenience, let us call such a structure a special quadratic structure on M and call the manifold equipped with such a structure a special quadratic manifold.

Observe that F reduces to an almost complex structure if α be set equal to zero in the above equation.

Theorem 1.1 — The rank of the special quadratic structure tensor is equal to the dimension of the manifold.

PROOF : Assuming $\bar{X} = 0$ we have $\bar{\bar{X}} = 0$. By virtue of (1*a*) and $\bar{X} = \bar{\bar{X}} = 0$, we obtain $X = 0$. That is: the kernel of F is the trivial subspace $\{0\}$ of $x(M)$. Hence nullity ν of F is 0. If ρ denote the rank of F then according to an important theorem in the theory of Linear Algebra (Halmos 1969) we shall have

$$\rho + \nu = n.$$

But $\nu = 0$. Therefore $\rho = n$.

Theorem 1.2 — The dimension n of the manifold M_n equipped with the special quadratic structure (1) ($\alpha^2 < 4$) is even.

PROOF : Let V be an eigen-vector corresponding to an eigenvalue λ. Therefore

$$\bar{V} = \lambda V$$

which yields

$$\bar{\bar{V}} = \lambda^2 V.$$

Substituting these values of $\bar{V}$ and $\bar{\bar{V}}$ in (1a) we obtain

$$\lambda^2 V + \alpha\lambda V + V = 0$$

which yields

$$\lambda^2 + \alpha\lambda + 1 = 0,$$

because V being an eigen vector is non-zero. The roots of the above quadratic equation are

$$\tfrac{1}{2}(-\alpha \pm \sqrt{\alpha^2 - 4}).$$

Let us consider the case when $\alpha^2 < 4$. The eigenvalues of F are $e^{\pm i\theta}$ where $\cos\theta = -\frac{\alpha}{2}$. Since complex eigenvalues occur in conjugate pairs therefore n is even.

Theorem 1.3 — The special quadratic structure defined on M_n is not unique.

PROOF : Let us set (Mishra 1973)

$$\mu(F'(X)) = F(\mu(X)) \qquad ...(2)$$

where F' stands for a tensor field of type (1, 1) and μ is a non-singular vector space homomorphism on $x(M)$. We have to show that F' is also a special quadratic structure. We have

$$\begin{aligned} \mu\left(F'^2(X) \right) &= \mu\left(F'(F'(X))\right) \\ &= F\left(\mu(F'(X))\right) \\ &= F\left(F(\mu(X))\right) \\ &= F^2(\mu(X)). \end{aligned}$$

Thus

$$\mu(F'^2(X) + \alpha F'(X) + X) = 0$$

which implies

$$F'^2(X) + \alpha F'(X) + X = 0$$

by virtue of non-singularity of μ. Hence F' is also a special quadratic structure.

Theorem 1.4 — Let F and F' be two special quadratic structures on the manifold M_n such that (2) holds. If V is an eigen-vector of F', $\mu(V)$ is an eigen-vector of F corresponding to the same eigenvalue.

PROOF : Since V is an eigen-vector of F', we therefore have

$$\mu(F'(V)) = \mu(\lambda V) = \lambda\mu(V).$$

Due to (2) the above equation becomes

$$F(\mu(V)) = \lambda\mu(V).$$

Hence λ is an eigenvalue of F and $\mu(V)$ is the corresponding eigen-vector, since μ is non-singular and V is non-zero.

Nijenhuis Tensor

The Nijenhuis tensor of the structure F is a vector valued bilinear skew-symmetric function N given by

$$N(X, Y) \overset{def}{=} [\bar{X}, \bar{Y}] + \overline{\overline{[X, Y]}} - \overline{[\bar{X}, Y]} - \overline{[X, \bar{Y}]}. \quad ...(3)$$

It is easy to show that

$$N(\bar{X}, Y) = N(X, \bar{Y}). \quad ...(4)$$

Barring X in (4) and using (1a) we have

$$N(\bar{\bar{X}}, \bar{Y}) = - N(X, Y) - \alpha N(\bar{X}, Y). \quad ...(5)$$

Barring (3) we obtain

$$\alpha\overline{N(X, Y)} = (1 - \alpha^2)\, N(X, Y) + N(\bar{X}, \bar{Y}). \quad ...(6)$$

On rendering α the value zero in (5) and (6) we come across

$$N(\bar{X}, \bar{Y}) = - N(X, Y)$$

which holds for the Nijenhuis tensor of the almost complex structure.

Theorem 1.5 — In order that a $2m$-dimensional differentiable manifold M_{2m} admits a special quadratic structure (1) ($\alpha^2 < 4$) it is necessary and sufficient that M_{2m} contains a distribution π_m of complex dimension m and a distribution $\bar{\pi}_m$ conjugate to π_m such that π_m and $\bar{\pi}_m$ have no direction in common and span together a vector space of dimension $2m$.

PROOF : Let M_{2m} be a special quadratic manifold with the quadratic structure F given by (1) ($\alpha^2 < 4$). Since F is real there are m eigenvalues $e^{i\theta}$ and m eigenvalues $e^{-i\theta}$. Corressponding to a multiple eigenvalue there will be a pencil of eigenvectors. Let $\{\underset{x}{P}\}_{(x=1,\ldots,m)}$ be the set of m linearly independent eigen-vectors corresponding to the eigenvalue $e^{i\theta}$ and let $\{\underset{x}{Q}\}_{(x=1,\ldots,m)}$ be the set of m linearly independent eigen-vectors corresponding to the eigenvalues $e^{-i\theta}$. Let $\underset{x}{Q}$ be the complex conjugate of $\underset{x}{P}$. We have to show that $\{\underset{x}{P}, \underset{x}{Q}\}$ is a linearly independent set. Let us assume that

$$\overset{x}{a}\underset{x}{P} + \overset{x}{b}\underset{x}{Q} = 0$$

which implies

$$\overset{x}{a}\underset{x}{\overline{P}} + \overset{x}{b}\underset{x}{\overline{Q}} = 0.$$

Putting the values of $\underset{x}{\overline{P}}$ and $\underset{x}{\overline{Q}}$, we have

$$\overset{x}{a}e^{i\theta}\underset{x}{P} + \overset{x}{b}e^{-i\theta}\underset{x}{Q} = 0.$$

Eliminating $\underset{x}{Q}$ from the above two equations, we get

$$(1 - e^{2i\theta})\,\overset{x}{a}\underset{x}{P} = 0$$

which implies that $\overset{x}{a} = 0$ because $\underset{x}{P}$ are linearly independent. Similarly we can prove that $\overset{x}{b} = 0$. Hence $\{\underset{x}{P}, \underset{x}{Q}\}$ is a linearly independent set.

Let L and M be two linear transformations defined by

$$L(X) = X - e^{i\theta}\bar{X} \qquad \ldots(7)$$

$$M(X) = X - e^{-i\theta}\bar{X}. \qquad \ldots(8)$$

Accordingly we have

$$L(\underset{x}{P}) = (1 - e^{2i\theta})\,\underset{x}{P} \qquad \ldots(9a)$$

$$L(\underset{x}{Q}) = 0 \qquad \ldots(9b)$$

$$M(\underset{x}{P}) = 0 \qquad \ldots(10a)$$

$$M(\underset{x}{Q}) = (1 - e^{-2i\theta})\,\underset{x}{Q}. \qquad \ldots(10b)$$

In this way we obtained two distributions π_m and $\tilde{\pi}_m$ complex conjugate to each other and of complex dimension m; such as to have no common direction, with projections L and M respectively.

Conversely, suppose that there occur a distribution π_m of complex dimension m and a distribution $\tilde{\pi}_m$ complex conjugate to π_m having no direction common with π_m and π_m together with $\tilde{\pi}_m$ span a linear manifold of dimension $2m$. Suppose that in the distribution π_m there are m linearly independent vectors $\underset{x}{P}$ and in the distribution $\tilde{\pi}_m$ there are m linearly independent vectors $\underset{x}{Q} \cdot \underset{x}{P}$ and $\underset{x}{Q}$ span a linear manifold of dimension $2m$. $\{\underset{x}{P}, \underset{x}{Q}\}$ is then linearly independent.

Let us construct the inverse set $\{\overset{x}{p}, \overset{x}{q}\}_{x=1, \ldots, m}$ of the set $\{\underset{x}{P}, \underset{x}{Q}\}$. Then

$$\overset{x}{p}(\underset{y}{P}) = \overset{x}{\underset{y}{\delta}}, \ \overset{x}{p}(\underset{y}{Q}) = 0 \qquad \ldots(11)$$

$$\overset{x}{q}(\underset{y}{P}) = 0, \ \overset{x}{q}(\underset{y}{Q}) = \overset{x}{\underset{y}{\delta}} \qquad \ldots(12)$$

and

$$\overset{x}{p}(X)\, \underset{x}{P} + \overset{x}{q}(X)\, \underset{x}{Q} = X. \qquad \ldots(13)$$

Further; setting

$$\bar{X} = e^{i\theta}\overset{x}{p}(X)\, \underset{x}{P} + e^{-i\theta}\overset{x}{q}(X)\, \underset{x}{Q}$$

we find

$$\bar{\bar{X}} = e^{i\theta}\, \overset{x}{p}(\bar{X})\, \underset{x}{P} + e^{-i\theta}\overset{x}{q}(\bar{X})\, \underset{x}{Q}$$

$$= e^{2i\theta}\, \overset{x}{p}(X)\, \underset{x}{P} + e^{-2i\theta}\overset{x}{q}(X)\, \underset{x}{Q},$$

to this we add (13) securing

$$\bar{\bar{X}} + X = 2(\cos\theta)\, \bar{X} = -\, \alpha\bar{X}$$

which skews that F defines a special quadratic structure.

Corollary 1.1 — We have

$$L^2 = (1 - e^{2i\theta})\, L \qquad \ldots(14)$$

$$M^2 = (1 - e^{-2i\theta})\, M \qquad \ldots(15)$$

and

$$LM = ML = 0. \qquad \ldots(16)$$

PROOF : Due to (7) we have

$$\begin{aligned} L(L(X)) &= L(X) - e^{i\theta}\overline{L(X)} \\ &= X - e^{i\theta}\bar{X} - e^{i\theta}(\overline{X - e^{i\theta}\bar{X}}) \\ &= X - e^{i\theta}\bar{X} - e^{i\theta}\bar{X} + e^{2i\theta}\bar{\bar{X}} \\ &= (1 - e^{2i\theta})\, X - e^{i\theta}(1 - e^{2i\theta})\, \bar{X} \\ &= (1 - e^{2i\theta})\, L(X). \end{aligned}$$

From (8) we get the second result.

Now

$$\begin{aligned} L(M(X)) &= M(X) - e^{i\theta}\,\overline{M(X)} \\ &= (X - e^{-i\theta}\bar{X}) - e^{i\theta}(X - e^{-i\theta}\bar{X}) = 0. \end{aligned}$$

Similarly we can show that $ML = 0$.

Lemma 1.1 — The necessary and sufficient condition that the distribution $\pi_m(\tilde{\pi}_m)$ be integrable is $(dM)\,(X, Y) = 0$ $((dL)\,(X, Y) = 0)$.

PROOF : Let π_m be integrable. Then

$$X, Y \in \pi_m \Rightarrow [X, Y] \in \pi_m$$

Hence

$$M(X) = 0, \quad M(Y) = 0 \quad \text{and} \quad M([X, Y]) = 0.$$

We have (Quan 1969)

$$(dM)\,(X, Y) = XM(Y) - YM(X) - M([X, Y]).$$

Due to the above fact it yields $(dM)\,(X, Y) = 0$.

Conversely; if $(dM)\,(X, Y) = 0$, let $X, Y \in \pi_m$. Then

$$M([X, Y]) = 0$$

because $M(X) = 0$ and $M(Y) = 0$. Also

$$\begin{aligned} L([X, Y]) = [X, Y] - e^{i\theta}\,\overline{[X, Y]} &= [X, Y] - e^{2i\theta}\,[X, Y] \\ &= (1 - e^{2i\theta})\,[X, Y] \end{aligned}$$

which proves that if $X, Y \in \pi_m$ then $[X, Y] \in \pi_m$. That is π_m is integrable. Similarly we can prove the second part of the lemma.

Theorem 1.6 — In order that a special quadratic structure of class C^∞ be completely integrable it is necessary and sufficient that the Nijenhuis tensor vanishes.

PROOF : The necessary and sufficient condition for the integrability of π_m is $(dM)\,(X, Y) = 0$. That is

$$(dM)\left\{\frac{1}{1-e^{2i\theta}}\,L(X),\ \frac{1}{1-e^{2i\theta}}\,L(Y)\right\} = 0$$

which yields

$$L(X)\,\{M(L(Y))\} - M(X)\,\{L(M(Y)\} - M([L(X), L(Y)]) = 0.$$

But $LM = ML = 0$, so the condition becomes

$$M([L(X), L(Y)]) = 0$$

which is equivalent to

$$[X, Y] + \overline{[\bar{X}, Y]} + \overline{[X, \bar{Y}]} + e^{2i\theta}\,[\bar{X}, \bar{Y}] - e^{-i\theta}\,\overline{[X, Y]}$$

$$- e^{i\theta}([\bar{X}, Y] + [X, \bar{Y}] + \overline{[\bar{X}, \bar{Y}]}) = 0. \qquad \ldots(17)$$

Similarly we can show that the necessary and sufficient condition that the distribution $\bar{\pi}_m$ be integrable is that

$$[X, Y] + \overline{[\bar{X}, Y]} + \overline{[X, \bar{Y}]} + e^{-2i\theta}\,[\bar{X}, \bar{Y}] - e^{i\theta}\,\overline{[X, Y]}$$

$$- e^{-i\theta}([\bar{X}, Y] + [X, \bar{Y}] + \overline{[\bar{X}, \bar{Y}]}) = 0. \qquad \ldots(18)$$

Substracting (18) from (17), we have

$$\alpha[\bar{X}, \bar{Y}] + [\bar{X}, Y] + [X, \bar{Y}] + \overline{[\bar{X}, \bar{Y}]} - \overline{[X, Y]} = 0.$$

Barring this equation and using ($1a$) and (3), we obtain $N = 0$.

Conversely, suppose $N = 0$. Then

$$\overline{[\bar{X}, Y]} + \overline{[X, \bar{Y}]} = [\bar{X}, \bar{Y}] + \overline{\overline{[X, Y]}}.$$

Barring it we have

$$[\bar{X}, Y] + [X, \bar{Y}] = \overline{[X, Y]} - \overline{[\bar{X}, \bar{Y}]} - \alpha\,[\bar{X}, \bar{Y}].$$

Substituting the values of $[\bar{X}, Y] + [X, \bar{Y}]$ and $\overline{[\bar{X}, Y]} + \overline{[X, \bar{Y}]}$ in the left-hand side of (17) and using ($1a$) and the value of α we observe that (17) is satisfied.

Hence $N = 0 \Rightarrow \pi_m$ is integrable. Similarly it can be shown that $N = 0 \Rightarrow \bar{\pi}_m$ is integrable. Therefore, $N = 0 \Rightarrow$ the special quadratic structure is integrable.

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