ON A STRUCTURE f SATISFYING $f^2 + \alpha f - I = 0$

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From the quadratic structure a structure has been defined in this paper; which is a generalization of an almost product structure. The existence and the integrability condition of this structure have been obtained.

1. INTRODUCTION

Let us consider a C^∞-manifold M_n of dimension n with a tensor-field f of type (1, 1) satisfying the algebraic equation (Yano 1963)

$$f^2 + \alpha f - I = 0 \ (\alpha \in R). \qquad ...(1.1)$$

If X denote an arbitrary vector-field in the $C^\infty(M)$-module $x(M)$ of M then (1.1) can be written in the form

$$\bar{\bar{X}} + \alpha \bar{X} - X = 0, \qquad ...(1.1a)$$

where $\bar{X}$ represents $f(X)$. The rank of f can be easily shown to be equal to n (Sinha and Sharma 1978).

Let λ be an eigen value of f. Then λ is a root of the quadratic equation (Mishra 1973)

$$\lambda^2 + \alpha\lambda - 1 = 0. \qquad ...(1.2)$$

The roots of (1.2) are

$$\tfrac{1}{2}(-\alpha \pm \sqrt{\alpha^2 + 4})$$

which may be taken to be e^θ and $-e^{-\theta}$ if we set $\sinh\theta$ for $-\frac{1}{2}\alpha$. Suppose that the multiplicities of e^θ and $-e^{-\theta}$ are p and $q(p + q = n)$ respectively.

It should be noted meanwhile that if α were a rational number and $\sqrt{\alpha^2 + 4}$ irrational; the roots of (1.2) would be surds and since surd-roots of an equation with rational coefficients occur in pairs; the dimension of the manifold would be even. Consequently $p = q = \frac{1}{2}n$.

2. DISTRIBUTIONS ON M_n WITH STRUCTURE (1.1)

Let us define a pair of projection operators l and m on the tangent space at each point of M_n by

$$l(X) = \frac{1}{1 + e^{2\theta}} (X + e^{\theta} \bar{X}), \qquad ...(2.1)$$

$$m(X) = \frac{1}{1 + e^{-2\theta}} (X - e^{-\theta} \bar{X}). \qquad ...(2.2)$$

Theorem 2.1 — We have the following relations :

(a) $l^2 = l$, (b) $lm = 0$, (c) $ml = 0$, (d) $m^2 = m$, (e) $l + m = I$. ...(2.3)

PROOF :

$$l^2(X) = l(l(X)) = \frac{1}{1 + e^{2\theta}} \left\{ l(X) + \frac{e^{\theta}}{1 + e^{2\theta}} (\bar{X} + e^{\theta} X - \alpha e^{\theta} \bar{X}) \right\}$$

$$= \frac{1}{1 + e^{2\theta}} \left\{ l(X) + \frac{e^{2\theta}}{1 + e^{2\theta}} (X + e^{\theta} \bar{X}) \right\}$$

$$= l(X).$$

Next, we have

$$l(m(X)) = \frac{1}{1 + e^{2\theta}} \left\{ m(X) + \frac{e^{\theta}}{1 + e^{-2\theta}} (\bar{X} - e^{-\theta} \bar{\bar{X}}) \right\}$$

$$= \frac{1}{1 + e^{2\theta}} \left\{ m(X) - \frac{e^{\theta}}{1 + e^{-2\theta}} (e^{-\theta} X - e^{-2\theta} \bar{X}) \right\}$$

$$= \frac{1}{1 + e^{2\theta}} \left\{ m(X) - \frac{1}{1 + e^{-2\theta}} (X - e^{-\theta} \bar{X}) \right\}$$

$$= 0.$$

Similarly the other relations can be proved.

These projections l and m applied to the tangent space at each point of the manifold are complementary projections. There exist two complementary distributions π_p and π_q of dimensions p and q corresponding to l and m respectively. Let $\{\underset{a}{P}\}$, $a = 1, ..., p$ and $\{\underset{x}{Q}\}$, $x = 1, ..., q$ be linearly independent sets of vectors, spanning π_p and π_q respectively. It can be shown that the set $\{\underset{a}{P}, \underset{x}{Q}\}$ forms a linearly independent set. Let $\{\overset{a}{p}, \overset{x}{q}\}$ be the inverse set of the set $\{\underset{a}{P}, \underset{x}{Q}\}$. Then

$$\overset{a}{p}(\underset{b}{P}) = \overset{a}{\underset{b}{\delta}}, \; \overset{a}{p}(\underset{x}{Q}) = 0, \qquad ...(2.4)$$

$$\overset{x}{q}(\underset{a}{P}) = 0, \; \overset{x}{q}(\underset{y}{Q}) = \overset{x}{\underset{y}{\delta}}, \qquad ...(2.5)$$

$$\overset{a}{p}(X) \underset{a}{P} + \overset{x}{q}(X) \underset{x}{Q} = X. \qquad ...(2.6)$$

The set $\{\underset{a}{P}, \underset{x}{Q}\}$ is termed as a non-holonomic frame for M_n.

Theorem 2.2 — We have

(a) $l(X) = \overset{a}{p}(X) \underset{a}{P}$

(b) $m(X) = \overset{x}{q}(X) \underset{x}{Q}$. ...(2.7)

PROOF : $l(X) = l\{\overset{a}{p}(X) \underset{a}{P} + \overset{x}{q}(X) \underset{x}{Q}\}$

$$= \overset{a}{p}(X) \underset{a}{P}.$$

Similarly we can get the second result.

Theorem 2.3 — A necessary and sufficient condition for M_n to admit the structure f satisfying $f^2 + \alpha f - I = 0$ is that the distributions π_p and π_q have no direction in common and they together span an n-dimensional linear space.

PROOF : Let M_n be equipped with a structure f satisfying (1.1). Then with the help of the projections l and m defined by (2.1) and (2.2), we get a pair of global distributions π_p and π_q having no direction in common. Moreover, if P and Q be linearly independent sets of vectors spanning π_p and π_q it is easy to show that the set $\{\underset{a}{P}, \underset{x}{Q}\}$ is a linearly independent set. Hence the distributions π_p and π_q together span a linear space of dimension n.

Now, let there occur a pair of distributions π_p and π_q having no direction in common and spanning a linear space of dimension n. Let $\{\underset{a}{P}\}$ be the set of p linearly independent vectors in π_p and $\{\underset{x}{Q}\}$ be the set of q linearly idependent vectors in π_q. $\{\underset{a}{P}, \underset{x}{Q}\}$ is a linearly independent set. Define the inverse set $\{\overset{a}{p}, \overset{x}{q}\}$ as in (2.4), (2.5) and (2.6). Set up

$$\bar{X} = e^{\theta}\overset{a}{p}(X) \underset{a}{P} - e^{-\theta}\overset{x}{q}(X) \underset{x}{Q}.$$

Therefore

$$\bar{\bar{X}} = e^{\theta}\overset{a}{p}\{e^{\theta}\overset{b}{p}(X) \underset{b}{P} - e^{-\theta}\overset{y}{q}(X) \underset{y}{Q}\} \underset{a}{P}$$

$$- e^{-\theta}\overset{x}{q}\{e^{\theta}\overset{b}{p}(X) \underset{b}{P} - e^{-\theta}\overset{y}{q}(X) \underset{y}{Q}\} \underset{x}{Q}$$

$$= e^{2\theta}\overset{a}{p}(X) \underset{a}{P} + e^{-2\theta}\overset{x}{q}(X) \underset{x}{Q},$$

whence

$$\bar{\bar{X}} + \alpha \bar{X} = (e^{2\theta} + \alpha e^{\theta})\, \overset{a}{p}(X) \underset{a}{P} + (e^{-2\theta} - \alpha e^{-\theta})\, \overset{x}{q}(X) \underset{x}{Q}$$

$$= e^{\theta}(e^{\theta} + e^{-\theta} - e^{\theta})\, \overset{a}{p}(X) \underset{a}{P} + e^{-\theta}(e^{-\theta} + e^{\theta} - e^{-\theta})\, \overset{x}{q}(X) \underset{x}{Q}$$

$$= \overset{a}{p}(X) \underset{a}{P} + \overset{x}{q}(X) \underset{x}{Q}$$

$$= X$$

which gives the structure (1.1).

3. Nijenhuis Tensor of f

The Nijenhuis tensor N of the structure f satisfying (1.1) is a skew-symmetric tensor of type (1, 2) given by (Yano 1965)

$$N(X, Y) = [\bar{X}, \bar{Y}] + \overline{\overline{[X, Y]}} - \overline{[\bar{X}, Y]} - \overline{[X, \bar{Y}]}. \qquad \text{...(3.1)}$$

Theorem 3.1 — We have

$$N(\bar{X}, Y) = N(X, \bar{Y}), \qquad \text{...(3.2)}$$

$$N(\bar{X}, \bar{Y}) = N(X, Y) - \alpha N(X, \bar{Y}) \qquad \text{...(3.3)}$$

$$N(\bar{X}, Y) = - \alpha N(X, Y) - \overline{N(X, Y)}. \qquad \text{...(3.4)}$$

Proof : $N(\bar{X}, Y) = [\bar{\bar{X}}, Y] + \overline{\overline{[\bar{X}, Y]}} - \overline{[\bar{\bar{X}}, Y]} - \overline{[\bar{X}, \bar{Y}]}$

which reduces to

$$N(\bar{X}, Y) = [\bar{X}, Y] + [X, \bar{Y}] - [\bar{X}, \bar{Y}] - \overline{[X, Y]} - \alpha\, [\bar{X}, \bar{Y}]. \qquad \text{...(3.5)}$$

Using skew-symmetry of $N(X, Y)$ in X and Y and interchanging X and Y in (3.5), we obtain (3.2). (3.3) follows on barring (3.2). Next, barring (3.1) we find

$$[\bar{X}, Y] + [X, \bar{Y}] - \overline{[X, Y]} - \overline{\overline{[\bar{X}, \bar{Y}]}} - \alpha\, [\bar{X}, \bar{Y}] = - \overline{N(X, Y)} - \alpha N(X, Y).$$

This reduces to (3.4) due to (3.5).

4. Integrability Condition

Lemma 4.1 — In order that the distribution $\pi_p(\pi_q)$ be integrable it is necessary and sufficient that

$$(dm)\,(X, Y) = 0((dl)\,(X, Y) = 0).$$

PROOF : First, assume that π_p is integrable. Then

$$X, Y \in \pi_p \Rightarrow [X, Y] \in \pi_p$$

Therefore,

$$l(X) = X, l(Y) = Y, \ m(X) = 0, \ m(Y) = 0 \text{ and } m([X, Y]) = 0.$$

Incorporating these in the definition (Quan 1969)

$$(dm)(X, Y) = Xm(Y) - Ym(X) - m([X, Y])$$

we obtain $(dm)(X, Y) = 0$.

Conversely, suppose that $(dm)(X, Y) = 0$, where $X, Y \in \pi_p\, m$. Then $([X, Y]) = 0$ because $m(X) = 0$ and $m(Y) = 0$. Moreover, $l([X, Y]) = \{[X, Y] + e^{\theta}\, \overline{[X, Y]}\}/1 + e^{2\theta}$, which, in view of

$$m([X, Y]) = \frac{1}{1 + e^{-2\theta}}\left([X, Y] - e^{-\theta}\, \overline{[X, Y]}\right) = 0$$

is equal to $[X, Y]$. Hence $[X, Y] \in \pi_p$. Thus it is proved that if $X, Y \in \pi_p$; $[X, Y] \in \pi_p$. That means π_p is integrable. Similarly the second part can be proved.

Theorem 4.1 — A necessary and sufficient condition for the structure f satisfying (1.1) to be integrable is that the Nijenhuis tensor of f vanishes identically.

PROOF : By lemma (4.1), the necessary and sufficient condition that π_p is integrable is $(dm)(X, Y) = 0$, where X and Y satisfy $l(X) = X$. The condition is equivalent to $(dm)(l(X), l(Y)) = 0$ or $m([l(X), l(Y)]) = 0$ because $ml = 0$. In the light of (2.1) and (2.2) it takes the form

$$[X + e^{\theta}\bar{X}, Y + e^{\theta}\bar{Y}] - e^{-\theta}\, \overline{[X + e^{\theta}\bar{X}, Y + e^{\theta}\bar{Y}]} = 0$$

equivalently

$$\overline{\overline{[X, Y]}} + e^{\theta}([\bar{X}, Y] + [X, \bar{Y}] - \overline{[X, Y]} - \overline{[\bar{X}, \bar{Y}]})$$
$$+ e^{2\theta}([\bar{X}, \bar{Y}]) - \overline{[\bar{X}, Y]} - \overline{[X, \bar{Y}]} = 0$$

which reduces, by virtue of (3.5) and (3.4), to

$$N(X, Y) + e^{\theta}\{-\overline{N(X, Y)} - \alpha N(X, Y)\} = 0$$

that is

$$N(X, Y) - e^{-\theta}\overline{N(X, Y)} = 0. \quad \ldots(4.1)$$

Proceeding on the similar lines and taking the help of lemma (4.1) we can show that the necessary and sufficient condition that the distribution π_q be completely integrable is that

$$N(X, Y) + e^{\theta}\overline{N(X, Y)} = 0. \quad \ldots(4.2)$$

Reconciliation of the conditions (4.1) and (4.2) enables us to claim that the necessary and sufficient condition for the complete integrability of both the distributions π_p and π_q is $N = 0$. Thus f is integrable if and only if $N = 0$.

Remark: The conditions (4.1) and (4.2) can also be looked upon as $m(N(X, Y)) = 0$ and $l(N(X, Y)) = 0$. That is, π_p is integrable if and only if m acts on $N(X, Y)$ as a null operator and π_q is integrable if and only if l acts on $N(X, Y)$ as a null operator.

REFERENCES

Mishra, R. S. (1973). A differentiable manifold with f-structure of rank r. *Tensor, N. S.*, **27**, 370-78.

Quan, P. H. (1969). Introduction à la géométrie des variétés différentiables. Dunod, Paris.

Sinha, B. B., and Sharma, R. (1978). On a special quadratic structure on differentiable manifolds. *Indian J. pure appl. Math.*, **9**.

Yano, K. (1963). On a structure defined by a tensor field of type (1, 1) satisfying $f^3 + f = 0$. *Tensor, N. S.*, **14**, 9-19.

———— (1965). Differential Geometry on Complex and Almost Complex Spaces. Pergamon Press, London.