PROPAGATION OF A PRESSURE SHOCK IN GASES AT HIGH TEMPERATURE

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The propagation of a pressure shock through a viscous and heat conducting gas has been studied by taking into account the effects of thermal radiation. The law of propagation of a pressure shock has been determined. It is found that the velocity of propagation of the shock decreases with the strength of the pressure shock. A differential equation governing the growth and decay of a pressure shock during propagation has been obtained. It is concluded that a pressure shock will decay and ultimately terminate into an accoustic disturbance.

INTRODUCTION

Recently in many new technological developments such as reentry of space vehicles, fission and fussion reactions and others, the temperature of gas is so high and the density of the gas is so low that the thermal radiation plays an important role in the determination of the flow field. Most of the current literature on radiation gas dynamics is concerned with only radiative heat flux effects (Pai 1966, Scala and Sampson 1964 and Pai and Taso 1966) and very little has been done about the effects of radiation energy density and radiation pressure. The object of the present investigation is to study the propagation of a pressure shock by taking into account the effects of thermal radiation on the growth and decay of the pressure shock. Since the gas is viscous, we can assume discrete discontinuities in pressure and temperature, while the velocity and density are continuous across the wave surface. Such a discontinuity is defined as a pressure shock (Thomas and Edstrom 1961). Pai (1966) has suggested that when the mean free path of radiation is very small, the radiative heat transfer term can be neglected. We assume a pressure discontinuity in an optically thick medium with such a high temperature and low density that the radiation pressure number R_p is not negligible, but the profiles structured by radiant heat transfer can be assumed to be imbedded in the discontinuity.

The following boundary conditions, which are found appropriate to this problem are (Thomas and Endstrom 1961) :

$$\text{(a)} \quad \overline{u}_i = 0; \quad \text{(b)} \quad \overline{\frac{dp}{dn}} = 0; \quad \text{(c)} \quad \overline{\frac{dT}{dt}} = 0. \qquad ...(1)$$

The bar appearing in the relations (1) denotes evaluation at the rear or flow side of the shock surface $\Sigma(t)$. The pressure condition in (1*b*) states that normal directional derivative of pressure vanishes on the flow side of $\Sigma(t)$. The temperature condition in (1*c*) implies that the temperature of a material particle has a stationary value at the instant of its contact with the rear of the surface $\Sigma(t)$. In view of the fact that the velocity vanishes on the surface $\Sigma(t)$, this condition can also be written as:

$$\overline{\frac{\partial T}{\partial t}} = 0. \qquad \ldots(2)$$

Basic Equations

The basic equations governing the gas flow under considerations are :

$$\frac{\partial \rho}{\partial t} + u_i \rho_{,i} + \rho u_{i,i} = 0 \qquad \ldots(3)$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j u_{i,j} - \sigma_{ij,j} = 0 \qquad \ldots(4)$$

$$\frac{\partial}{\partial t}(\rho e + \tfrac{1}{2}\rho u^2 + E^R) + \frac{\partial}{\partial x^k}\{u_k(\rho e + \tfrac{1}{2}\rho u^2 + E^R + P + P^R)\}$$
$$- \frac{\partial u_i \sigma_{ij}}{\partial x^j} - K \frac{\partial}{\partial x^k}\left(\frac{\partial T}{\partial x^k}\right) = 0 \qquad \ldots(5)$$

where

$$\sigma_{ij} = -(p + p^R)\,\delta_{ij} - \tfrac{2}{3}\mu u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \qquad \ldots(6)$$

and u_i, p, ρ, T, e are respectively the velocity components, pressure density, absolute temperature and internal energy of the gas. A comma followed by an index (i) denotes partial differentiation with respect to the corresponding coordinate.

The thermal equation of state for a radiating gas is given by

$$p = \rho R T. \qquad \ldots(7)$$

Since the gas is optically thick, the Rosseland approximation may be used for radiation pressure and radiation energy density and thus we have

$$E^R = 3p^R = a_R T^4 \qquad \ldots(8)$$

where p^R, E^R and a_R respectively represent the radiation pressure, the radiation energy density and the Stefan-Boltzmann constant.

The geometrical and kinematical conditions of first order for the present case are (Thomas 1957):

$$[u_{i,j}] = \overline{u_{i,j}} = \lambda_i n_j, \qquad \ldots(9a)$$

$$\left[\frac{\partial u_i}{\partial t}\right] = \frac{\overline{\partial u_i}}{\partial t} = -\lambda_i G, \qquad ...(9b)$$

$$[\rho_{,i}] = \bar{\rho}_{,i}\, \zeta n_i, \qquad ...(10a)$$

$$\left[\frac{\partial \rho}{\partial t}\right] = \frac{\overline{\partial \rho}}{\partial t} = -\zeta G, \qquad ...(10b)$$

where $\lambda_i = [u_{i,j}]\, n_j$ and $\zeta = [\rho_{,j}]\, n_j$ are scalar functions defined over the surface $\Sigma(t)$. The bracket $[f]$ denotes the jump in the quantity enclosed at contiguous points on the singular surface $\Sigma(t)$. The Rankine-Hugonist shock relations are :

$$\rho(u_n - G) = \bar{\rho}(\bar{u}_n - G) \qquad ...(11)$$

$$[\sigma_{ij}]\, n_j = \bar{\rho}(\bar{u}_n - G)\, [u_i] \qquad ...(12)$$

$$[\sigma_{ij} u_j]\, n_i + K\, [T_{,i}]\, n_i = \bar{\rho}(u_n - G)\, [E] \qquad ...(13)$$

where

$$[E] = C_V\, [T] + \frac{3}{\rho}\, [p^R] + \tfrac{1}{2}[u_i \,.\, u_i].$$

Using (6), (12) and (9*a*), we obtain

$$[p + p^R]\, n_i + \tfrac{2}{3}\, \mu \lambda_k n_k n_i = \mu \lambda_i + \mu \lambda_k n_k n_i. \qquad ...(14)$$

Now we represent the shock surface $\Sigma(t)$ parameterically by function $x_i = x_i(u^1, u^2, t)$. In consequence of (14), we can write

$$\lambda = \lambda n_i, \qquad ...(15)$$

where λ is a scalar function defined over the surface $\Sigma(t)$.

From eqns. (14) and (15), we get

$$[p + p^R] = \tfrac{4}{3}\, \mu \lambda. \qquad ...(16)$$

Taking jump in (7), we have

$$[p] = \rho R [T]. \qquad ...(17)$$

In view of the relation (8) and (17), eqn. (16) can be written in the form

$$\eta = \frac{4\mu\lambda}{3p_0} - p_0^R \{(\eta + 1)^4 - 1\}, \qquad ...(18)$$

where $\eta = \dfrac{[p]}{p_0}$, p_0 and p_0^R are the values of pressure and radiation pressure infront of the pressure shock.

Making use of eqns. (13), (16) and (17), we obtain

$$[T_{,i}]\, n_i = -\frac{G}{RK}\{C_V (4 - 3\gamma)\, \eta p_0 + 4R\mu\lambda\}. \qquad \text{...(19)}$$

Making use of (9), (10) in (3), we get

$$\zeta G = \rho\lambda. \qquad \text{...(20)}$$

In consequence of the boundary conditions (1*b*) and (1*c*), it is now readily seen that the compatibility conditions of first order for the pressure and temperature can be put in the following forms :

$$[p_{,i}] = \tfrac{4}{3}\mu g^{\alpha\beta}\lambda_{,\alpha} x_{i,\beta} \qquad \text{...(21a)}$$

$$\left[\frac{\partial}{\partial t}(p + p^R)\right] = \tfrac{4}{3}\mu\frac{\delta\lambda}{\delta t} \qquad \text{...(21b)}$$

$$[T_{,i}] = \overline{T_{,i}} = \bar{T}_{,j} n_j n_i + [T]\, g^{\alpha\beta}\lambda_{,\alpha} x_{i,\beta} \qquad \text{...(22a)}$$

$$\frac{\partial [T]}{\partial t} = G\, [T_{,i}]\, n_i \qquad \text{...(22b)}$$

where $g^{\alpha\beta}$ are the contravariant components of the fundamental metric tensor of the surface $\Sigma(t)$ and $x_{i,\beta}$ are the components of the projective tensor. The $\lambda_{,\alpha}$ are the surface derivatives of λ and $\frac{\delta\lambda}{\delta t}$ is the material derivative as observed from the wave front.

Differentiating (7) with respect to x_i, we get

$$[p_{,i}] = R\rho\, [T_{,i}] + R\, [\rho_{,i} T] \qquad \text{...(23)}$$

$$[\rho_{,i} T] = [\rho_{,i}]\, [T] + T_0\, [\rho_{,i}]. \qquad \text{...(24)}$$

Multiplying (23) by n_i and summing on the repeated index i the resulting equation yields

$$G^2 = \frac{K(\eta + 1) p_0 \lambda R}{C_V (4 - 3\gamma) p_0 \eta + 4\mu\lambda R}, \qquad \text{...(25)}$$

Using (18) in (25), we get

$$G^2 = \frac{3C_0^2 (1 + \eta)\{1 + R_{p_0}(\eta^3 + 4\eta^2 + 6\eta + 4)\}}{4P_r\{1 + 3(\gamma - 1) R_{p_0}(\eta^3 + 4\eta^2 + 6\eta + 4)\}}, \qquad \text{...(26)}$$

where $P_r = \frac{\mu C_p}{K}$ is the well known Prandtle number.

For non-radiating gas ($R_{P_0} = 0$), the relation (26) assumes the form

$$G^2 = \frac{3\gamma}{4P_r}(p_0 + \tfrac{4}{3}\mu\lambda), \qquad \text{...(27)}$$

which is in full agreement with the result of Thomas and Edstrom (1961).

The relation (26) shows that the velocity of propagation will decrease with the strength η of the pressure shock.

For weak shocks $\eta = 0$ and therefore, for a weak shock, we have

$$\tfrac{4}{3} P_r \frac{G^2}{C_0^2} = \delta^2 = \frac{1 + 4R_{P_0}}{1 + 3(\gamma - 1)R_{P_0}}$$

which shows that a pressure shock will ultimately degenerate into a small disturbance propagating with the effective speed of sound given by above relation.

Growth and Decay of a Pressure Shock

Taking the δ-time derivative of (18) we get

$$\frac{\delta\eta}{\delta t}\{1 + 4R_{P_0}(\eta + 1)^3\} = \tfrac{4}{3}\frac{\mu}{p_0}\frac{\delta\lambda}{\delta t}. \qquad \text{...(28)}$$

Using (17) and (19) in (22*b*), we get

$$p_0 \frac{\delta\eta}{\delta t} = \rho G^2 \{C_V(4 - 3\gamma)\eta p_0 + 4\mu\lambda\rho R\} \qquad \text{...(29)}$$

From (28) and (29), we obtain

$$\tfrac{4}{3}\mu\frac{\delta\lambda}{\delta t} = \rho G^2 \{C_V(4 - 3\gamma)\eta p_0 + 4\mu\lambda\rho R\}\{1 + 4R_{P_0}(1 + \eta)^3\}. \qquad \text{...(30)}$$

Substituting G^2 from (26) in (30), we get

$$\frac{\delta\eta}{\delta t} = \frac{3\gamma C_V \eta}{4P_r K}(\eta + 1)\{1 + R_{P_0}(\eta^3 + 4\eta^2 + 6\eta + 4\} p_0. \qquad \text{...(31)}$$

Let $\Sigma(t_0)$ represents the position of the wave-surface at time $t = t_0$ and let σ denote the distance measured from $\Sigma(t_0)$ along the normal trajectory to the family of surface $\Sigma(t)$ in the direction of propagation. The discontinuity η can be regarded as a function of the distance σ along the normal trajectory and hence we have

$$\frac{\delta\eta}{\delta t} = G\frac{d\eta}{d\sigma} \qquad \text{...(32)}$$

Making use of (32) and (26) in (31), we get

$$\frac{d\eta}{d\Gamma} = -\eta(1 + \eta)^{1/2}\{1 + R_{P_0}(\eta^3 + 4\eta^2 + 6\eta + 4)\}^{1/2}$$
$$\times \{1 + 3(\gamma - 1)(\eta^3 + 4\eta^2 + 6\eta + 4) R_{P_0}\}^{1/2}, \qquad \text{...(33)}$$

where

$$\Gamma = \sqrt{\tfrac{3}{4}} P_r \frac{p_0}{C_0 \mu} \sigma.$$

For $R_{P_0} = 0$, eqn. (33) takes the form

$$\frac{d\eta}{d\Gamma} = -\eta(1 + \eta)^{1/2}. \qquad \text{...(34)}$$

Solving (34), we obtain the solution for η in the form :

$$(1 + \eta)^{1/2} = \frac{1 + De^{-\Gamma}}{1 + De^{-\Gamma}} \qquad \text{...(35)}$$

where D is the constant of integration. If we suppose that the initial pressure discontinuity $\eta_0 = 1$ at $\Gamma = 0$ the solution (35) can be put into the form

$$\eta = \frac{4}{(3 + 2\sqrt{2})\, e^{\Gamma} + (3 - 2\sqrt{2})\, e^{-\Gamma} - 2}$$

which clearly shows that $\eta \to 0$ as $\Gamma \to \infty$. This is in excellent agreement with the result of Thomas and Edstrom (1961) for an ordinary gas. It is also concluded that a pressure shock will decay and ultimately terminates into an accoustic disturbances.

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