UNSTEADY FLOW PAST A FLAT PLATE BY FINITE DIFFERENCE METHOD

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(*Received* 7 *January* 1978; *after revision* 24 *April* 1978)

The problem of unsteady laminar flow past a flat plate has been solved by finite difference technique. Uniform suction, that follows step function change, is applied normal to the plate. Numerical results are obtained when the suction velocity doubles in the step change and velocity profiles have been drawn. The method yields quite satisfactory results, with the accuracy equal to the square of the time step.

INTRODUCTION

A number of problems concerning flow with suction can be found in the literature, as the latter provides a way for controlling the boundary layer. Schillichting (1960) obtained an exact solution to the problem of uniform flow past a flat plate at zero incidence, subject to uniform suction. Later Stuart (1955) extended the same problem by considering the flow, fluctuating periodically about a mean value. Watson (1958) generalized Stuart's (1955) solution for the case when the flow is a general function of time. However, Kelley (1965) investigated the effect of time dependent suction and observed that the Laplace Transform technique becomes quite complicated when suction is arbitrary function of time.

In the present note, the finite difference method has been employed to solve the non-linear partial differential equation, governing the flow when the suction follows a step function change. The stability condition has already been satisfied. Fairly accurate results have been obtained within a few steps.

GOVERNING EQUATIONS

Consider a two dimensional flow of a viscous incompressible fluid past a porous flat plate in the plane $y = 0$. The axes x and y are chosen to be along and normal to the plate. Let u and v be the velocity components along x and y axes respectively. A constant suction velocity $v_1 < 0$, normal to the plate is applied for time $t < 0$. At $t = 0$, the normal velocity is changed to $v_2 < 0$ and is maintained for all times $t > 0$. The fluid has a constant velocity U_∞ parallel to the plate for all time.

The momentum and continuity equation of the flow are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad \ldots(1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad \ldots(2)$$

Assuming the velocity to be independent of x, (1) can be written as

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \qquad \ldots(3)$$

The boundary conditions are

$$\left.\begin{aligned} u(0, t) &= 0, \\ u(\infty, t) &= U_\infty, \\ v(0, t) &= v_1, \ t \leqslant 0 \\ &= v_2, \ t > 0 \end{aligned}\right\} \qquad \ldots(4)$$

and the initial condition is

$$u(y, 0) = U_\infty\left(1 - \exp\left(\frac{v_1 y}{\nu}\right)\right). \qquad \ldots(5)$$

Introducing the dimensionless variables,

$$\eta = \frac{|v_1|\,y}{\nu}, \quad \tau = \frac{|v_1|^2\,t}{\nu}, \quad \bar{u} = \frac{u}{U_\infty} \text{ and } \bar{v} = \frac{v}{U_\infty} \qquad \ldots(6)$$

equation (5) is transformed to

$$\frac{\partial \bar{u}}{\partial \tau} + \frac{\bar{v}}{|v_1|}\left(\frac{\partial \bar{u}}{\partial \eta}\right) = \frac{\partial^2 \bar{u}}{\partial \eta^2}. \qquad \ldots(7)$$

The corresponding boundary conditions are

$$\left.\begin{aligned} \bar{u}(0, \tau) &= 0 \\ \bar{u}(\infty, \tau) &= 1 \\ \frac{\bar{v}}{|v_1|} &= -1, \ \tau \leqslant 0 \\ &= -\lambda, \ \tau > 0 \end{aligned}\right\} \qquad \ldots(8)$$

and the initial condition is

$$\bar{u}(\eta, 0) = 1 - e^{-\eta}. \qquad \ldots(9)$$

The numerical value of λ has been taken as 2 in further analysis. However, the method will work for any value of λ.

METHOD OF SOLUTION

Defining a new dimensionless variable,

$$\eta' = \frac{\eta}{1+\eta}, \qquad \text{...(10)}$$

the governing equation takes the form,

$$\frac{\partial \bar{u}}{\partial \tau} - 2\eta'(1-\eta')^2 \frac{\partial \bar{u}}{\partial \eta'} = (1-\eta')^4 \frac{\partial^2 \bar{u}}{\partial \eta'^2} \qquad \text{...(11)}$$

along with

$$\left.\begin{aligned} &\bar{u}(0,\tau) = 0, \\ &\bar{u}(1,\tau) = 1, \\ \text{and} \quad &\bar{u}\left(\frac{\eta'}{1-\eta'}, 0\right) = 1 - e^{\eta'/(1-\eta')}. \end{aligned}\right\} \qquad \text{...(12)}$$

For convenience, dropping dashes and bars, eqn. (11) appears to be,

$$\frac{\partial u}{\partial \tau} - 2\eta(1-\eta)^2 \frac{\partial u}{\partial \eta} = (1-\eta)^4 \frac{\partial^2 u}{\partial \eta^2} \qquad \text{...(13)}$$

with

$$\left.\begin{aligned} &u(0,\tau) = 0 \\ &u(1,\tau) = 1 \\ \text{and} \quad &u\left(\frac{\eta}{1-\eta}, 0\right) = 1 - e^{-\eta/(1-\eta)}. \end{aligned}\right\} \qquad \text{...(14)}$$

FINITE DIFFERENCE EQUATIONS

The explicit method of finite difference technique has been employed here. The flow region is defined as a semi infinite strip bounded by $\eta = 0$ and $\eta = 1$. To construct the difference equation, the region of interest is divided into a grid or mesh of lines parallel to η and τ axis. The spacing between the lines parallel to η-axis is chosen while spacing between the lines parallel to τ-axis is taken so as to satisfy the stability condition according to Ralston (1960). Fig. 1 illustrates the construction of grid. Solution of the difference equation is obtained at the intersection of these grid lines, called nodes. These nodes are specified by the double subscript (i, j), with the origin (0, 1) located at the intersection of the plane $\eta = 0$, and $\tau = 0$. All the values of the dependent variable u at the nodal points along the plane $\eta = 0$ are given by $u(0, \tau)$ which are known while unknown values $u^*_{i,j}$ are to be determined. The following difference scheme is made :

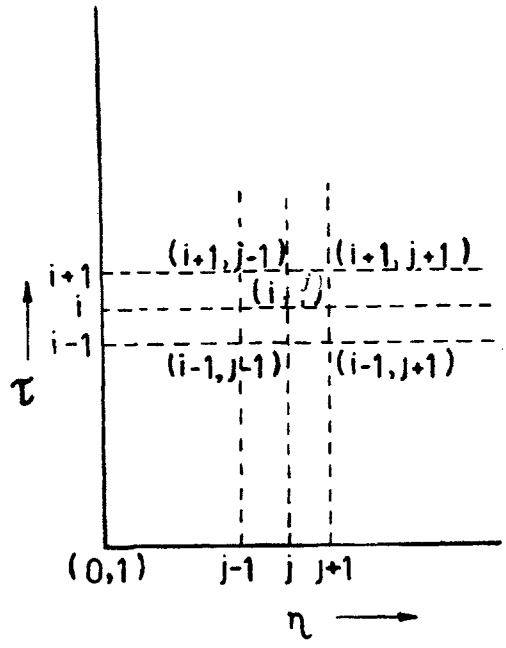

FIG. 1.

$$\left.\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{u^*_{i,j} - u_{i,j}}{\Delta \tau} \\ \frac{\partial u}{\partial \eta} &= \frac{u_{i,j} - u_{i,j-1}}{\Delta \eta} \\ \frac{\partial^2 u}{\partial \eta^2} &= \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta \eta)^2}. \end{aligned}\right\} \quad \ldots(15)$$

Substituting these values into eqn. (13), we get

$$u^*_{i,j} = u_{i,j} + (1 - \eta_{i,j})^2 \frac{\Delta \tau}{\Delta \eta} \Big[(1 - \eta_{i,j})^2 \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta \eta} + 2\eta_{i,j}(u_{i,j} - u_{i,j-1}) \Big]. \quad \ldots(16)$$

The boundary conditions (14) now, are

$$\left.\begin{aligned} u_{0,j} &= 1 - \exp\,(-\,(j-1)\,\Delta\eta/1 - (j-1)\,\Delta\eta),\ j = 2, \ldots, 10 \\ u_{i,1} &= 0 \\ u_{i,11} &= 1.0. \end{aligned}\right\} \quad \ldots(17)$$

Here in eqn. (16), during any time step, $u_{i,j-1}$ are assumed to be known whereas $u_{i,j+1}$ are treated equal to $u_{i-1,j+1}$.

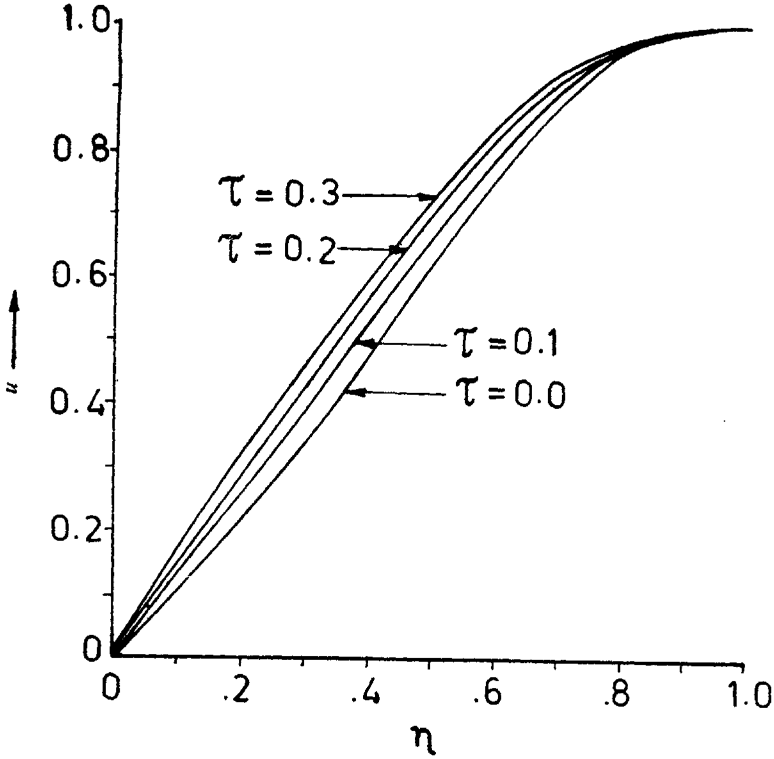

FIG. 2. Variation of u with η for different τ.

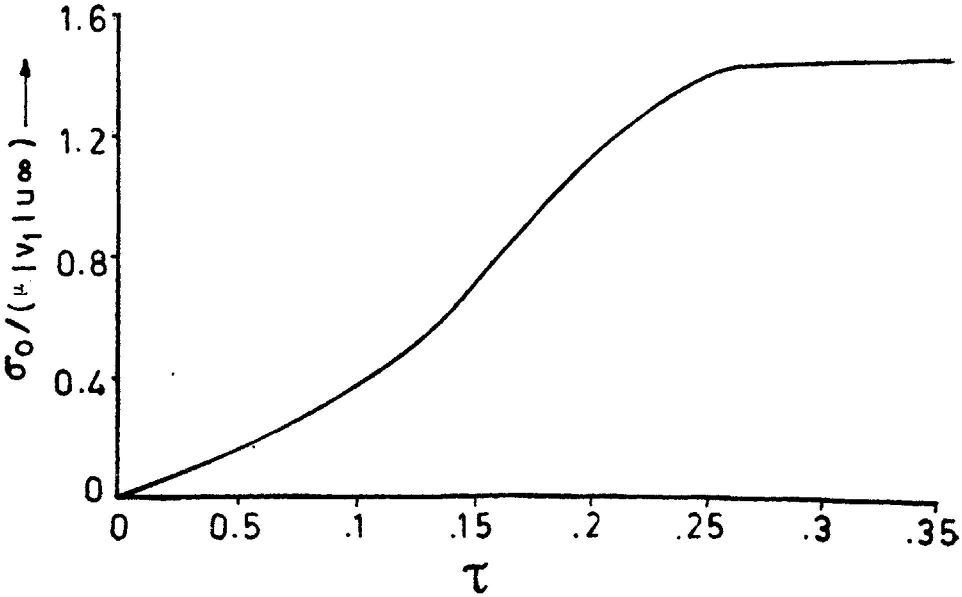

FIG. 3. Variation of $\sigma_0/(\mu \mid v_1 \mid U_\infty)$ with τ.

Discussion of Results

In the calculations the value of $\triangle\eta$ has been chosen to be 0.1 while $\triangle\tau$, as per stability condition is taken as 0.005. Velocity curves has been plotted for different values of $\tau = 0.0$, 0.1, 0.2 and 0.3 in Fig. 2. It reveals that the velocity was uniformly to a constant value as $\eta \to 1$. Also the increase in τ is followed by an increase in velocity reaching to the same constant value near $\eta = 1$.

The difference scheme is consistent as per condition stated in Ralston (1960) with the given differential eqn. (13) so as to ensure that the difference equations actually do 'approximate' the differential equation. The truncation error for the approximation is $O(\triangle\tau)^2 + O(\triangle\eta)$ which tends to zero with $\triangle\eta$ and $\triangle\tau$ both tending to zero. Moreover convergence of the scheme is implied, as for a sufficiently fine mesh, the numerical solution is in close approximation to the exact solution of the differential equation, obtained by Purohit and Goyal (1975) using Laplace transform.

The shearing stress at the wall is given by

$$\sigma = \mu\left(\frac{\partial u}{\partial y}\right)_0$$

$$= \mu \mid v_1 \mid U_\infty \left(\frac{\partial u}{\partial \eta}\right)_{\eta=0} \qquad \ldots(18)$$

$\sigma_0/(\mu \mid v_1 \mid U_\infty)$ has been calculated by using forward difference formula and has been shown in Fig. 3. It is easy to note that the shearing stress on the wall increases with τ and attains a constant value after $\tau = 0.25$.

Acknowledgement

The authors are grateful to Prof. C. Prasad for providing necessary facilities in carrying out the work.

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