SOME NEW SEQUENCE SPACES

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It is natural to expect that almost convergence must be related to some concept of almost bounded variation in the same vein as convergence is related to bounded variation. The purpose of this paper is to investigate this new concept in some details. Some inclusion theorems have been established and some matrix transformations have been characterized.

§1. Let s be the set of all sequences real or complex and l_∞, c and c_0 respectively be the Banach spaces of bounded, convergent and null sequences $x = \{x_n\}_{n=0}^{\infty}$ normed by $\| x \| = \sup_{n \geqslant 0} | x_n |$. Let D be the shift operator on S, that is,

$$Dx = \{x_n\}_{n=1}^{\infty}, \quad D^2 x = \{x_n\}_{n=2}^{\infty}$$

and so on. It is evident that D is a bounded linear operator on l_∞ onto itself and that $\| D^k \| = 1 (\forall\, k)$.

It may be recalled that Banach Limit L (*see* Banach 1932) is a non-negative linear functional on l_∞ such that L is invariant under the shift operator that is, $L(Dx) = L(x)\ \forall\ x \in l_\infty$ and that $L(e) = 1$ where $e = \{1, 1, ...\}$. A sequence $x \in l_\infty$ is called almost convergent (*see* Lorentz 1948) if all Banach limits of x coincide. Let $\hat{c}$ denote the set of all almost convergent sequences.

It is natural to accept that almost convergence must be related to some concept $\widehat{BV}$ in the same vein as convergence is related to the concept of BV. BV denotes the set of all sequences of bounded variation and a sequence in $\widehat{BV}$ will mean a sequence of almost bounded variation.

The main object of this paper is to study this new concept in some detail. Also a new sequence space $\widehat{\boldsymbol{BV}}$ which is apparently more general than $\widehat{BV}$ naturally comes up for investigation and is considered along with $\widehat{BV}$.

We may remark here that the concept $\hat{l}$ of absolute almost convergence (which emerges naturally as the absolute analogue of almost convergence) and a new sequence space $\hat{\hat{l}}$ have been recently introduced (*see* Kuttner *et al.* 1978) and investigated.

§2. Consider the sequences of bounded linear transformations $d_{m,n} : l_\infty \to l_\infty$ defined by

$$d_{m,n}(x) = \frac{1}{m+1} \sum_{i=0}^{m} D^i x_n \qquad \ldots(2.1)$$

with $D^0 = 1$. It is evident that

$$d_{0,n}(x) = x_n = D^0 x_n \qquad \ldots(2.2)$$

and that

$$\| d_{m,n} \| = 1 \quad (\forall\ m, n).$$

Lorentz (1948) proved that

$$\hat{c} = \{x : \lim_{m\to\infty} d_{m,n}(x) \text{ exists uniformly in } n\}.$$

Now define

$$d_{-1,n}(x) = D^{-1} x_n = x_{n-1} \qquad \ldots(2.3)$$

and then write for $m, n \geqslant 0$,

$$t_{m,n}(x) = d_{m,n}(x) - d_{m-1,n}(x). \qquad \ldots(2.4)$$

So that by (2.2), (2.3) and (2.4) we have

$$t_{0,n}(x) = D^0 x_n - D^{-1} x_n = x_n - x_{n-1}. \qquad \ldots(2.5)$$

When $m \geqslant 1$ a straightforward calculation shows that

$$t_{m,n}(x) = \frac{1}{m(m+1)} \sum_{\nu=1}^{m} \nu(x_{n+\nu} - x_{n+\nu-1}).$$

Now we write

$$\hat{BV} = \{x : \sum_m | t_{m,n}(x) | \text{ converges uniformly in } n\},$$

and

$$\hat{\hat{BV}} = \{x : \sup_n \sum_m | t_{m,n}(x) | < \infty\}.$$

Here and afterwards summation without limits runs from 0 to ∞. Note that

$$BV = \{x : \sum_k | x_k - x_{k-1} | < \infty\}$$

where we define $x_{-1} = 0$. BV is a Banach space normed by

$$\| x \| = \sum_k | x_k - x_{k-1} | .$$

Given an infinite series

$$\Sigma a_n$$

we write

$$x_n = a_0 + \ldots + a_n.$$

Then we put

$$t_{m,n}(x) = \frac{1}{m(m+1)} \sum_{\nu=1}^{m} \nu a_{n+\nu} = \varphi_{m,n}(a).$$

So that by (2.5)

$$\varphi_{0,n}(a) = t_{0,n}(x) = a_n.$$

We define (*see* Kuttner *et al.* 1978).

$$\hat{l} = \{a : \sum_m | \varphi_{m,n}(a) | \text{ converges uniformly in } n\}$$

and

$$\hat{\hat{l}} = \{a : \sup_n \sum_m | \varphi_{m,n}(a) | < \infty\}.$$

§3. *The spaces* $\hat{BV}$ *and* $\hat{\hat{BV}}$ — We have the following theorem :

Theorem 1 — $\hat{BV} \subset \hat{\hat{BV}}$.

PROOF : Suppose that $x \in \hat{BV}$ and write $t_{m,n}$ for $t_{m,n}(x)$. We have to show that $\sum_m | t_{m,n} |$ is bounded. By the definition of $\hat{BV}$ $\exists$ an integer M such that

$$\sum_{m \geqslant M} | t_{m,n} | \leqslant 1 \ (\forall\, n)$$

Therefore it follows that for $m \geqslant M$,

$$| t_{m,n} | \leqslant 1 \ (\forall\, n).$$

It is now enough to show that, for fixed m, $t_{m,n}$ is bounded in n. Let $m \geqslant 1$ be fixed. A straightforward calculation shows that

$$x_{n+m} - x_{n+m-1} = (m+1)\, t_{m,n} - (m-1)\, t_{m-1,n}.$$

Hence for any fixed $m \geqslant M + 1$, $x_\nu - x_{\nu-1}$ is bounded and so $t_{m,n}$ is bounded for all m and n.

This completes the proof.

Remark : It is now a pertinent question, whether $\hat{\hat{BV}} \subset \hat{BV}$, that is, whether $\hat{BV} = \hat{\hat{BV}}$. We are not able to answer this question and it remains open.

Theorem 2 — $\hat{l} \subset \hat{BV} \subset \hat{c}$ and both the inclusions are proper.

PROOF : We have

$$\sum_m | t_{m,n}(y) | = \sum_m \frac{1}{m(m+1)} \left| \sum_{\nu=1}^{m} \nu y_{n+\nu} - \sum_{\nu=1}^{m} \nu y_{n+\nu-1} \right|$$

$$\leqslant \sum_m \frac{1}{m(m+1)} \left| \sum_{\nu=1}^{m} \nu y_{n+\nu} \right| + \sum_m \frac{1}{m(m+1)} \times \left| \sum_{\nu=1}^{m} \nu y_{n+\nu-1} \right|$$

$$= \sum_m | \varphi_{m,n}(y) | + \sum_m | \varphi_{m,n-1}(y) | . \qquad \ldots(3.1)$$

Let $y \in \hat{l}$. Then from the inequality (3.1) it is clear that $y \in \hat{BV}$. To see that the inclusion is proper consider the sequence $e = \{1, 1, \ldots\}$. Clearly $e \notin \hat{l}$ but $e \in \hat{BV}$.

Now $y \in \hat{BV}$ clearly implies the uniform convergence of

$$\sum_m t_{m,n}(y) \qquad \ldots(3.2)$$

with respect to n. But from (2.4) it is evident that uniform convergence of (3.2) implies uniform convergence of $d_{m,n}(y)$ with respect to n. Therefore it follows that $y \in \hat{c}$, that is, $\widehat{BV} \subset \hat{c}$. To see that the inclusion is proper consider the sequence $y = \{1, 0, 1, 0, \ldots\}$.

In this case $y + Dy = e$. Therefore

$$\begin{aligned} L(y) &= \tfrac{1}{2}\{L(y) + L(Dy)\} \\ &= \tfrac{1}{2} L(e) = \tfrac{1}{2}. \end{aligned}$$

Thus $y \in \hat{c}$. But simple calculation shows that

$$t_{m,n}(y) = \begin{cases} -1/2m \ (m, n \text{ odd}) \\ 1/2m \ (m \text{ odd}, n \text{ even}) \end{cases}$$

and

$$t_{m,n}(y) = \begin{cases} -1/2(m+1) \ (m, n \text{ even}) \\ 1/2(m+1) \ (m \text{ even}, n \text{ odd}). \end{cases}$$

Thus

$$\sum_m | t_{m,n}(y) | = \infty \ (\forall\, n),$$

and so that $y \notin \widehat{BV}$.

This completes the proof.

Theorem 3 — $\widehat{BV}$ is a Banach space normed by

$$\| x \|_\wedge = \sup_{n \geqslant 0} \sum_m | t_{m,n}(x) | \qquad \ldots(3.3)$$

PROOF : Because of Theorem 1, (3.3) is meaningful. It is a routine verification that $\widehat{BV}$ is a normed linear space. To show that $\widehat{BV}$ is complete in its norm topology, let $\{x^i\}_{i=0}^\infty$ be a Cauchy sequence in $\widehat{BV}$. Then $\{x_n^i\}_{i=0}^\infty$ is a Cauchy sequence in $\mathbb{C}$ for each n. Therefore $x_n^i \to x_n$ (say). Put $x = \{x_n\}_{n=0}^\infty$. We now show that $x \in \widehat{BV}$ and $\| x^i - x \| \to 0$. Since $\{x\}^i$ is a Cauchy sequence in $\widehat{BV}$, given $\in > 0 \ \exists\, N$ such that for $i, j > N$;

$$\sum_m | t_{m,n}(x^i - x^j) | < \in \ (\forall\, n)$$

Therefore for any M and $i, j > N$;

$$\sum_{m=0}^{M} | t_{m,n}(x^i - x^j) | < \in (\forall n).$$

Now taking limit as $j \to \infty$ and then as $M \to \infty$ we get for $i > N$;

$$\sum_m | t_{m,n}(x^i - x) | \leqslant \in (\forall n). \qquad ...(3.4)$$

Thus $x^i - x \in \overset{\wedge}{BV}$ and therefore by linearity $x \in \overset{\wedge}{BV}$. Also (3.4) implies that

$$\| x^i - x \| \leqslant \in (i > N).$$

This completes the proof.

Theorem 4 — $BV \subset \overset{\wedge}{BV}$ and the inclusion is proper. Further $x \in BV$ implies that

$$\| x \|_\wedge \leqslant \| x \|. \qquad ...(3.5)$$

For the proof of Theorem 4 we require the following result.

Lemma (*see* Maddox (1970, p. 168) — Let $(a_{m,n})$ be an arbitrary matrix of complex numbers such that

$$\sum_m | a_{mn} | < \infty \text{ for each } n$$

and

$$\sum_m | a_{mn} | \to 0 \text{ as } n \to \infty,$$

then

$$\sum_m | a_{mn} | \text{ converges uniformly in } n.$$

PROOF OF THEOREM 4 : For $m \geqslant 1$ and $n > 0$,

$$| t_{m,n}(x) | = \frac{1}{m(m+1)} \left| \sum_{\nu=1}^{m} \nu(x_{n+\nu} - x_{n+\nu-1}) \right|$$

$$\leqslant \frac{1}{m(m+1)} \sum_{\nu=1}^{m} \nu \, | x_{n+\nu} - x_{n+\nu-1} | .$$

Therefore

$$\sum_m | t_{m,n}(x) | \leqslant \sum_{\nu=1}^{\infty} \nu \, | x_{n+\nu} - x_{n+\nu-1} | \sum_{m=\nu}^{\infty} \frac{1}{m(m+1)}$$

$$\leqslant \sum_{\nu=1}^{\infty} | x_{n+\nu} - x_{n+\nu-1} |$$

and since $t_{0,n}(x) = x_n - x_{n-1}$ we have

$$\sum_m | t_{m,n}(x) | \leqslant \sum_{\nu=0}^{\infty} | x_{n+\nu} - x_{n+\nu-1} |$$

$$= \sum_{\nu=n}^{\infty} | x_\nu - x_{\nu-1} | . \qquad \ldots(3.6)$$

Now let $x \in BV$. Then

$$\sum_{\nu=n}^{\infty} | x_\nu - x_{\nu-1} | < \infty \quad \text{(for each } n\text{)}$$

and $\to 0$ as $n \to \infty$. Hence by the Lemma,

$\sum_m | t_{m,n}(x) |$ converges uniformly in n.

Thus $x \in \widehat{BV}$. Now (3.5) follows from the inequality (3.6).

To see that the inclusion is proper consider the sequence $x = (x_n)_{n=0}^{\infty}$ such that $x_0 = 0$ and for $n \geqslant 1$

$$x_n - x_{n-1} = \frac{(-1)^n}{n}.$$

Clearly $x \notin BV$. Now for $m \geqslant 1$,

$$t_{m,n}(x) = \frac{1}{m(m+1)} \sum_{\rho=n+1}^{n+m} \frac{(-1)^\rho (\rho - n)}{\rho} \qquad \ldots(3.7)$$

Since $\frac{z-n}{z}$ is a nondecreasing function of z, the sum in (3.7) does not, in modulus, exceed the modulus of its last term. So that

$$| t_{m,n}(x) | \leqslant \frac{1}{(m+n)(m+1)} \leqslant \frac{1}{m(m+1)}$$

Thus $x \in \widehat{BV}$.

This completes the proof.

We now comeback to $\widehat{\widehat{BV}}$ and prove the following :

Theorem 5 — (i) $\widehat{\widehat{BV}}$ is a Banach space by the norm given in (3.3). Also we have (ii) $\widehat{BV}$ is a closed subspace of $\widehat{\widehat{BV}}$, and (iii) $\widehat{\widehat{l}} \subset \widehat{\widehat{BV}}$ and the inclusion is proper.

PROOF : Proof of (i) is same as that of Theorem 3. Since by definition $\widehat{BV}$ and $\widehat{\widehat{BV}}$ have the same norm and since $\widehat{BV}$ is complete, (ii) holds. (iii) follows from the inequality (3.1) of Theorem 2 by taking supremum with respect to n on both the sides.

§4. Finally we consider certain matrix transformations in $\widehat{BV}$.

Let X and Y be any two nonempty subsets of the space s of complex sequences and $A = (a_{nk})$ be an infinite matrix of complex numbers. We write $Ax = \{An(x)\}$ if $An(x) = \sum_k a_{nk} x_k$ converges for each n. If $x = (x_k) \in X \Rightarrow Ax = \{An(x)\} \in Y$ we say that A defines a matrix transformation from X into Y and we denote it by $A : X \rightarrow Y$. By (X, Y) we mean the class of matrices A such that $A : X \rightarrow Y$. Further if in X and Y there is any notion of limit or sum we write (X, Y, P) to denote the subset of (X, Y) which preserves the limit or sum.

We now characterise the matrices in the classes $(BV, \widehat{BV})$ and $(l_\infty, \widehat{BV})$. For this we write

$$t_{m,n}(Ax) = \frac{1}{m(m+1)} \sum_{\nu=1}^{m} \nu(A_{n+\nu}(x) - A_{n+\nu-1}(x))$$

$$= \sum_k a(n, k, m)\, x_k$$

where

$$a(n, k, m) = \frac{1}{m(m+1)} \sum_{\nu=1}^{m} \nu(a_{n+\nu,k} - a_{n+\nu-1,k}).$$

We have

Theorem 6 — $A \in (BV, \widehat{BV})$ if and only if

(i) $\exists$ a constant K :

$$\sum_m \left| \sum_{k=0}^{r} a(n, k, m) \right| \leqslant K (r, n = 1, 2, \ldots),$$

(ii) $\lim_{m\to\infty} a(n, k, m) = \alpha_k$ uniformly in n,

(iii) $\lim_{m\to\infty} \sum_k a(n, k, m) = \alpha$ uniformly in n.

PROOF : *Necessity* — Suppose that $A \in (BV, \hat{BV})$, that is, $\sum_m | t_{m,n}(Ax) |$ converges uniformly in n for every $x \in BV$. Write

$$q_n(x) = \sum_m | t_{m,n}(Ax) | .$$

Note that $\{q_n\}$ is a sequence of continuous seminorms on BV such that $\sup_n q_n(x) < \infty$. Therefore by Banach-Steinhaus Theorem (for example, *see* Maddox 1970, p. 114) $\exists$ a constant $M > 0$ such that

$$q_n(x) \leqslant M \| x \| \quad (\forall n)$$

Taking $x \in e$ we get

$$\sum_m \left| \sum_k a(n, k, m) \right| \leqslant M.$$

Now

$$\sum_m \left| \sum_{k=0}^{r} a(n, k, m) \right|$$

$$\leqslant \sum_m \left| \sum_{k=0}^{\infty} a(n, k, m) \right| + \sum_m \left| \sum_{k=r+1}^{\infty} a(n, k, m) \right|$$

$$\leqslant M + M = 2M.$$

Since $(BV, \hat{BV}) \subset (BV, \hat{c})$ (*see* Nanda 1974), (ii) and (iii) must hold.

This completes the necessity.

Sufficiency — Suppose that the conditions (i) – (iii) hold and that $x \in BV$. Fix $n \in Z^+$. Since $x \in BV$, $x_k \to l$. Now

$$\sum_m | t_{m,n}(Ax) |$$

$$\leqslant \sum_m \left| \sum_{k=0}^{r} a(n, k, m) \right| \sum_k | x_k - x_{k-1} | + l \left| \sum_k a(n, m, m) \right|$$

$$\leqslant K \| x \| + Kl.$$

Now $Ax \in \hat{BV}$ and consequently $Ax \in \hat{c}$. Therefore, we have (*see* Nanda 1974),

$$\lim_{m\to\infty} t_{m,n}(Ax) = l\alpha + \sum_k \alpha_k(x_k - l), \text{ uniformly in } n.$$

This completes the proof.

Corollary 1 — $A \in (BV, \hat{BV}, P)$ if and only if

(i) Condition (i) of Theorem 6 holds,

(ii) $\lim_{m\to\infty} a(n, k, m) = 0$ uniformly in n,

(iii) $\lim_{m\to\infty} \sum_k a(n, k, m) = 1$ uniformly in n.

PROOF : This follows from Theorem 6.

Theorem 7 — $A \in (l_\infty, \hat{BV})$ if and only if

(i) $\exists$ a constant $K > 0$ such that

$$\sum_m \sum_k | a(n, k, m) | \leqslant K \quad (\forall\, n),$$

(ii) $\lim_{m\to\infty} a(n, k, m) = \alpha_k$ uniformly in n,

(iii) $\lim_{m\to\infty} \sum_k | a(n, k, m) - \alpha_k | = 0$ uniformly in n.

PROOF : *Necessity* — Suppose that $A \in (l_\infty, \hat{BV})$. Write

$$q_n(x) = \sum_m | t_{m,n}(Ax) | .$$

Now $\{q_n\}$ is a sequence of continuous seminorms on l_∞ such that $\sup_n q_n(x) < \infty$ $\forall\, x \in l_\infty$. Therefore by Banach-Steinhaus Theorem $\exists$ a constant $K > 0$ such that

$$q_n(x) \leqslant K \| x \| \ (\forall\, x \in l_\infty).$$

Putting $x = \text{sgn}\, a(n, k, m)$ we observe that the condition (i) holds. Since $(l_\infty, \hat{BV}) \subset (l_\infty, \hat{c})$ (*see* Nanda 1974), the conditions (ii) and (iii) hold.

Sufficiency — Suppose that the conditions (i) – (iii) hold and that $x \in l_\infty$. Now

$$\sum_m | t_{m,n}(Ax) |$$

$$\leqslant \sum_m \sum_k | a(n, k, m) | (\sup_k | x_k |)$$

$$\leqslant K \| x \|.$$

Now $Ax \in \hat{BV}$. Consequently $Ax \in \hat{c}$ and therefore (*see* Nanda 1974),

$$\lim_{m \to \infty} \sum_k a(n, k, m) x_k = \sum_k \alpha_k x_k,$$

uniformly in n.

This completes the proof.

Corollary 2 — $(BV, \hat{BV}, P) \cap (l_\infty, \hat{BV}) = \phi$.

PROOF : This follows from Corollaries 1 and Theorem 7.

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