DUSTY VISCOUS FLOW BETWEEN OSCILLATING COAXIAL CIRCULAR CYLINDERS

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The motion of a dusty, viscous, incompressible fluid contained between two coaxial circular cylinders when the cylinders perform longitudinal oscillations along their coaxis, is considered. Three cases (i) when the inner cylinder oscillates while the outer cylinder moves with uniform velocity and (ii) when the outer cylinder oscillates while the inner cylinder moves with uniform velocity and (iii) when both the cylinders oscillate, are discussed. The velocity fields for the fluid and dust particles have been found and the skin friction drag acting on either cylinder is calculated. Particular cases of interest are deduced.

1. INTRODUCTION

The study of dusty viscous fluids has recently attracted a number of workers as there occur a variety of situations in which the motion of dusty fluids is involved. Such situations arise, for instance. in the movement of dust laden air, in fluidization, in the use of dust in gas cooling system and in sedimentation in tidal waves.

Using the formulation of Saffman (1962), who gave the equations governing the dusty viscous fluid motion, several authors gave exact solutions for various problems. Michael (1965) considered the Kelvin-Helmholtz instability of a dusty gas, Michael and Miller (1966) dealt with the plane parallel flow problem of a dusty gas. Michael and Norey (1968), Sambasiva Rao (1969), Verma and Mathur (1973) Tewari and Bhattacharjee (1972) and Kishore and Pandey (1977) studied the problem of circular cylinders under various conditions. The steady motion of a sphere in a dusty gas was considered by Sambasiva Rao (1973).

In this paper, we consider the flow of a dusty viscous fluid contained between two coaxial circular cylinders which perform longitudinal oscillations along their co-axis. The following three cases have been discussed : (i) When the inner cylinder performs oscillation while the outer cylinder moves with uniform velocity (ii) when the outer cylinder performs oscillations while the inner cylinder moves with uniform velocity and (iii) when both the cylinders perform oscillations with same phase. The velocity fields of the fluid and dust particles have been determined and the skin friction drag calculated. Particular cases of interest have been deduced.

2. Governing Equations

The equations governing the unsteady motion of a dusty, viscous, incompressible fluid as given by Saffman (1962) are

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} . \nabla) \vec{u} = - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} + \frac{KN}{\rho} (\vec{v} - \vec{u}) \qquad \text{...(2.1)}$$

$$m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} . \nabla) \vec{v} \right] = K(\vec{u} - \vec{v}) \qquad \text{...(2.2)}$$

$$\text{div } \vec{u} = 0 \qquad \text{...(2.3)}$$

$$\frac{\partial N}{\partial t} + \text{div} (N \vec{v}) = 0 \qquad \text{...(2.4)}$$

where $\vec{u}$ and $\vec{v}$ denote the local velocity vectors of the fluid and dust particles respectively, ρ the fluid density, p the static fluid pressure, ν the kinematic viscosity, N the number density of dust particles, K the Stokes resistance coefficient (for spherical particle of radius ϵ it is $6\pi\mu\epsilon$), μ the fluid viscosity and m the mass of a dust particle.

Let the polar cylindrical coordinates (r, ϕ, z) be taken so that z-axis lies along the common axis of the cylinders. Since the tube is very long, the field variables are independent of z and on account of the symmetry they are independent of ϕ and $u_\phi = 0$. Therefore, we denote the components of velocity of the fluid and those of the dust particles in (r, ϕ, z) directions by (u_1, u_2, u_3) and (u_1^*, u_2^*, u_3^*) respectively and we have

$$u_r = u_1(r, t), \; u_\phi = u_2 = 0, \; u_z = u_3(r, t) \qquad \text{...(2.5)}$$

$$u_r^* = u_1^*(r, t), \; u_\phi^* = u_2^* = 0, \; u_z = u_3^* (r, t).$$

Let us assume N to be constant. The equations of continuity (2.3) – (2.4) yield

$$u_1 = 0 \quad u_1^* = 0.$$

In the case of uniform pressure, eqns. (2.1) – (2.4) reduce to

$$\frac{\partial u_3}{\partial t} = \nu \left[\frac{\partial^2 u_3}{\partial r^2} + \frac{1}{r} \frac{\partial u_3}{\partial r} \right] + \frac{KN}{\rho} (u_3^* - u_3) \qquad \text{...(2.6)}$$

$$m \frac{\partial u_3^*}{\partial t} = K(u_3 - u_3^*). \qquad \text{...(2.7)}$$

We now introduce non-dimensional quantities :

$$\bar{t} = nt,\ \bar{u}_3 = \frac{u_3}{U},\ \bar{u}_3^* = \frac{u_3^*}{U},\ \bar{r} = \frac{r}{a}$$

$$f = \frac{mN}{\rho},\ g = \frac{\nu}{na^2},\ h = \frac{KN}{\rho n},\ n = \frac{b}{a}.$$

Here, U denotes the amplitude of the longitudinal oscillations and a and b denote the radii of the cylinders $0 \leqslant a < b$ in case (i) when n will be greater than unity; and $0 \leqslant b < a$ in case (ii) so that n is a positive fraction)

Equations (2.6) – (2.7) are transformed to

$$\frac{\partial \bar{u}_3}{\partial \bar{t}} = g\left(\frac{\partial^2 \bar{u}_3}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{u}_3}{\partial \bar{r}}\right) + h(\bar{u}_3^* - \bar{u}_3) \qquad ...(2.8)$$

$$\frac{\partial \bar{u}_3^*}{\partial \bar{t}} = \left(\frac{h}{f}\right)(\bar{u}_3 - \bar{u}_3^*). \qquad ...(2.9)$$

Eliminating $\bar{u}_3^*$ from (2.8) with the help of (2.9), we obtain

$$\frac{\partial^2 \bar{u}_3}{\partial \bar{t}^2} + c\frac{\partial \bar{u}_3}{\partial \bar{t}} = g\left(\frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\right)\left(\frac{\partial \bar{u}_3}{\partial \bar{t}} + e\bar{u}_3\right) \qquad ...(2.10)$$

where we have written

$$c = h(1 + f^{-1}),\quad e = \frac{h}{f}.$$

Now, we drop bars and suffixes for the sake of simplicity and write eqns. (2.9) and (2.10) as

$$\frac{\partial u^*}{\partial t} = e\,(u - u^*) \qquad ...(2.11)$$

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} = g\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial u}{\partial t} + eu\right) \qquad ...(2.12)$$

We have to solve eqns. (2.11) – (2.12) under appropriate boundary conditions. In a general unsteady problem, we specify the initial conditions when time $t = 0$. The three cases under study need only boundary conditions to be specified.

3. The Problem and Its Solution

Let us put

$$\left.\begin{aligned} u^* &= v_0^*(r) + v^*(r, t) \\ u &= v_0(r) + v(r, t) \end{aligned}\right\} \qquad ...(3.1)$$

where (v_0, v_0^*) and (v, v^*) represent the steady state and unsteady state solutions respectively. Consequently, we obtain for the fluid motion from (2.12),

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) v_0 = 0 \qquad \text{...(3.2)}$$

and

$$\frac{\partial^2 v}{\partial t^2} + c\frac{\partial v}{\partial t} = g\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial v}{\partial t} + ev\right) \qquad \text{...(3.2}a\text{)}$$

and for the dust particle motion

$$v_0^* = v_0 \qquad \text{...(3.3)}$$

$$\frac{\partial v^*}{\partial t} = e(v - v^*). \qquad \text{...(3.3}a\text{)}$$

Case (*i*) — Let the inner cylinder $r = 1$ perform longitudinal oscillations while the outer cylinder $r = n$ moves with uniform speed along their common axis. The boundary conditions for this case are

$$\left.\begin{aligned} &v_0 = 0, \;\; v = e^{it} \;\; \text{for } r = 1 \\ &v_0 = 1, \;\; v = 0 \;\;\;\; \text{for } r = n \; (n > 1). \end{aligned}\right\} \qquad \text{...(3.4)}$$

Solving (3.2) under the above boundary conditions we have

$$v_0^* = v_0 = \frac{\log r}{\log n}. \qquad \text{...(3.5)}$$

To solve (3.3), we substitute

$$v^* = \phi^*(r)\, e^{it},$$

$$v = \phi(r)\, e^{it}, \qquad \text{...(3.6)}$$

and have

$$\phi^* = \frac{e}{i + e}\,\phi$$

$$\phi'' + \frac{1}{r}\,\phi' - k^2\phi = 0 \qquad \text{...(3.7)}$$

where

$$k^2 = \frac{ci - 1}{g(i + e)} = \frac{c - e + (ce + 1)\, i}{g(1 + e^2)}. \qquad \text{...(3.8)}$$

Solution of (3.7) may be put as

$$\phi = AI_0(kr) + BK_0(kr)$$

where I_0 and K_0 are the modified Bessel functions of order zero and of first and second kind respectively. Consequently,

$$v(r, t) = e^{it} [AI_0(kr) + BK_0(kr)]. \quad ...(3.9)$$

The boundary conditions (3.4) yield

$$A = D^{-1}K_0(kn), \; B = - D^{-1}I_0(kn)$$

where

$$D = \begin{vmatrix} I_0(k) & I_0(kn) \\ K_0(k) & K_0(kn) \end{vmatrix}.$$

The solution may now be put as

$$v(r, t) = \frac{e^{it}}{D} \begin{vmatrix} I_0(kr) & I_0(kn) \\ K_0(kr) & K_0(kn) \end{vmatrix}. \quad ...(3.10)$$

We, therefore, have the velocity of the fluid and that of dust particles given by

$$\begin{aligned} u(r, t) &= v_0(r) + v(r, t) \\ &= \frac{\log r}{\log n} + \frac{I_0(kr)\, K_0(kn) - I_0(kn)\, K_0(kr)}{I_0(k)\, K_0(kn) - I_0(kn)\, K_0(k)}\, e^{it} \end{aligned} \quad ...(3.11)$$

and

$$\begin{aligned} u^*(r, t) &= v_0^*(r) + v^*(r, t) \\ &= \frac{\log r}{\log n} + \frac{ee^{it}}{i + e} \frac{I_0(kr)\, K_0(kn) - I_0(kn)\, K_0(kr)}{I_0(k)\, K_0(kn) - I_0(kn)\, K_0(k)}. \end{aligned} \quad ...(3.12)$$

Case (ii) — Let the outer cylinder be performing longitudinal oscillations while the inner cylinder moves uniformly along the coaxis (positive z-axis). In this case, we interchange a and b so that now

$0 \leqslant b < a$ and $b/a = n$ (a positive proper fraction).

The boundary conditions (3.4) with the above modification for n apply to the present case too and hence the solutions (3.10) - (3.11).

Particular case

If the inner cylinder is stationary we have

$$v_0 = v_0^* = 0$$

and the velocity fields are now given by

$$u(r, t) = e^{it} \frac{I_0(kr)\, K_0(kn) - I_0(kn)\, K_0(kr)}{I_0(k)\, K_0(kn) - I_0(kn)\, K_0(k)} \quad ...(3.13)$$

$$u^*(r, t) = \frac{ee^{it}}{i+e}\,\frac{I_0(kr)\,K_0(kn) - I_0(kn)\,K_0(kr)}{I_0(k)\,K_0(kn) - I_0(kn)\,K_0(k)}. \quad \ldots(3.14)$$

We can put (3.1) in the form

$$u(r, t) = e^{it}\,\frac{I_0(kr) - \dfrac{I_0(kn)}{K_0(kn)}\,K_0(kr)}{I_0(k) - \dfrac{I_0(kn)}{K_0(kn)}\,K_0(k)} \quad \ldots(3.15)$$

As the radius of the inner cylinder tends to zero, $\frac{b}{a} = n \rightarrow 0$ so that we have, for the oscillation of a single cylinder

$$u(r, t) = e^{it}\,\frac{I_0(kr)}{I_0(k)} \quad \ldots(3.16)$$

$$u^*(r, t) = \frac{ee^{it}}{i+e}\,\frac{I_0(kr)}{I_0(k)} \quad \ldots(3.17)$$

Here, we have used the fact (Lebedev 1965, p. 136) that as x tends to zero, $K_0(x)$ tends to infinity while $I_0(0) = 1$.

Case (iii) — If both the cylinders oscillate along their co-axis, we have to solve (2.11) - (2.12) under the boundary conditions

$$u = e^{it} \text{ for } r = 1$$

$$u = \epsilon e^{it} \text{ for } r = n\,(n > 1). \quad \ldots(3.18)$$

Making substitutions

$$u^* = \psi^*(r)\,e^{it}$$

$$u = \psi(r)\,e^{it}$$

we obtain from (2.11) - (2.12), the velocity of the fluid and that of the dust particles as

$$u = \frac{e^{it}}{D}\begin{vmatrix} I_0(kr) & I_0(kn) & I_0(k) \\ K_0(kr) & K_0(kn) & K_0(k) \\ 0 & \epsilon & 1 \end{vmatrix} \quad \ldots(3.19)$$

$$u^* = \frac{ee^{it}}{D(i+e)}\begin{vmatrix} I_0(kr) & I_0(kn) & I_0(k) \\ K_0(kr) & K_0(kn) & K_0(k) \\ 0 & \epsilon & 1 \end{vmatrix} \quad \ldots(3.20)$$

It can be readily seen that (3.9) can be got from (3.19) on putting $\epsilon = 0$, $t = 0$.

4. Drag

The formula for the skin friction drag per unit length on the inner circular cylinder $r = 1$ is

$$D = \mu \int_0^{2\pi} \left(r \frac{\partial u}{\partial r} \right)_{r=1} d\theta = 2\pi\mu \left(\frac{\partial u}{\partial r} \right)_{r=1} \qquad \text{...(4.1)}$$

In the case of oscillation of the inner cylinder while the outer one moves uniformly along z-axis, the drag calculated using eqn. (3.10), is

$$D_{1,t} = 2\pi\mu k \left[\frac{I'_0(k)\, K'_0(kn) - I_0(kn)\, K'_0(k)}{I_0(k)\, K_0(kn) - I_0(kn)\, K_0(k)}\, e^{it} + \frac{1}{\log n} \right] \qquad \text{...(4.2)}$$

and the drag on the outer cylinder $r = n$ is

$$D_{n,t} = 2\pi\mu k \left[e^{it} \frac{I'_0(kn)\, K_0(kn) - I_0(kn)\, K'_0(kn)}{I_0(k)\, K_0(kn) - I_0(kn)\, K_0(k)} + \frac{1}{\log n} \right] \qquad \text{...(4.3)}$$

where the primes on I_0 and K_0 indicate differentiation with respect to the argument kr.

Further, if both the cylinders move with uniform velocity, we deduce the corresponding expressions for drag from (4.2) – (4.3) as

$$D_{1,0} = 2\pi\mu k \,.\, \frac{I'_0(k)\, K_0(kn) - I_0(kn)\, K'_0(k)}{I_0(k)\, K_0(kn) - I_0(kn)\, K_0(k)} \qquad \text{...(4.4)}$$

$$D_{n,0} = 2\pi\mu k \,.\, \frac{I'_0(kn)\, K_0(kn) - I_0(kn)\, K'_0(kn)}{I_0(k)\, K_0(kn) - I_0(kn)\, K_0(k)} \qquad \text{...(4.5)}$$

by making the period of oscillation tend to infinity.

In the case of a single cylinder $r = 1$ oscillating with fluid inside it, the drag, which is obtained using (3.15), is

$$D_s = 2\pi\mu k\, e^{it} \frac{I'_0(k)}{I_0(k)}. \qquad \text{...(4.6)}$$

5. Deductions

When $|k|$ is small, we obtain from (3.15) the velocity field for the oscillation of single cylinder,

$$u(r, t) = e^{it} [\, 1 + \tfrac{1}{4} k^2(r^2 - 1)]$$

or

$$u(r, t) = \cos t + \frac{(r^2 - 1)}{4g(1 + e^2)} [(c - e) \cos t - (ce + 1) \sin t] \qquad \text{...(5.1)}$$

where we have written the real part only; and the expression for drag, from (4.6) reduces to

$$D_s = \pi\mu k^2 e^{it}$$

or
$$D_s = \frac{\pi\mu}{g(1 + e^2)} [(c - e) \cos t - (ce + 1) \sin t] \qquad ...(5.2)$$

taking only the real part.

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