ON FLUCTUATING FLOW OF AN ELASTICO-VISCOUS FLUID PAST AN INFINITE PLATE WITH VARIABLE SUCTION IN SLIP-FLOW REGIME

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An exact solution is obtained for the two dimensional flow of an elastico-viscous (Kuvshinski type) incompressible fluid past an infinite porous wall in slip-flow regime under the following conditions : (i) the suction velocity normal to the plate oscillates in magnitude but not in direction about a non-zero mean; (ii) the free-stream velocity oscillates in time about a constant mean. A particular case, when the suction velocity is constant, is also studied in detail. In slip flow regime the velocity in the boundary layer is more than that of non-rarefied fluids (no-slip conditions). For the same value of the elastic parameter λ, an increase in the rarefaction parameter h_1 leads to a decrease in the phase and amplitude of the skin-friction, both in case of constant and variable suction.

INTRODUCTION

Interest in the fluctuating flow of a viscous incompressible fluid past an infinite or semi-infinite flat plate has been shown by many researchers. First of all Lighthill (1954) initiated an important class of two dimensional time dependent flow problems dealing with the response of the boundary layer to external unsteady fluctuations about a mean value. Stuart (1955) found some interesting features of an oscillating flow past an infinite flat plate with constant suction. Reddy (1964) extended the work of Stuart (1955) by introducing the slip flow boundary conditions in place of no slip boundary conditions. Stuart's problem was later studied by Messiha (1966) for the case of variable suction. Suryaprakasrao (1962, 1963) generalized Stuart's problem to account for the effects of a magnetic field. All the aspects of velocity and temperature field in the case of the flow of incompressible, viscous and electrically conducting or non-conducting, Newtonian fluids have been studied.

Many common liquids such as oils, certain paints, polymer solutions, some organic liquids and many new materials of industrial importance exhibit both viscous and elastic properties. Therefore, the above fluids, called visco-elastic fluids, are also being studied extensively. Recently Dubey and Bhattacharya (1974) discussed the fluctuating flow of a visco-elastic fluid past an infinite flat plate with uniform suction.

In the present era of high altitude flights the study of fluids in slip flow regime has been recognized to be of immense importance. In case of flow in slip flow regime, the ordinary continuum approach fails to yield satisfactory results. Results agreeing with the observed physical phenomena can be obtained by solving the usual Navier-Stokes equations together with modified boundary conditions. In the present paper an attempt has been made to study the fluctuating flow of a visco-elastic fluid past an infinite flat plate subjected to time dependent suction in slip flow regime. The effects of elasticity, variable suction velocity and rarefaction parameter on the skin friction phase, amplitude and the velocity field are discussed by means of some graphs. The variable suction velocity has been taken as $v'_0(1 + \epsilon A \exp(in' t'))$ with the external flow velocity taken as $U'_0(1 + \epsilon \exp(in' t'))$.

Mathematical Analysis

The constitutive equations characterizing the elastico-viscous fluid considered here are

$$\left(1 + \lambda_0 \frac{D}{Dt}\right) p_{ik} = 2\mu e_{ik} \qquad \ldots(1)$$

where

$$\frac{D}{Dt}(p_{ij}) = \frac{\partial p_{ij}}{\partial t} + \frac{v_m \partial p_{ij}}{\partial x_m} \qquad \ldots(2)$$

$$e_{ij} = \tfrac{1}{2}\left[\left(\frac{\partial v_i}{\partial x_j}\right) + \left(\frac{\partial v_j}{\partial x_i}\right)\right] \qquad \ldots(3)$$

$$S_{ij} = - p g_{ij} + p_{ij}. \qquad \ldots(4)$$

p is the static pressure, g_{ij} the associated metric tensor and p_{ij} a tensor usually related to the rate of strain, e_{ij}, by the equation of state (1).

A coordinate system is chosen with the wall lying along the positive x'-axis and y'-axis being perpendicular to it. Under these conditions, the flow is independent of the distance parallel to the wall and the suction velocity v', normal to the wall, is directed towards it. Under these assumptions the relevant equations governing the flow and the equation of continuity are

$$\rho'\left(\frac{\partial u'}{\partial t'} + v'\frac{\partial u'}{\partial y'}\right) = -\frac{\partial p'}{\partial x'} + \frac{\partial p'_{x'y'}}{\partial y'} \qquad \ldots(5)$$

$$\rho'\frac{\partial v'}{\partial t'} = -\frac{\partial p'}{\partial y'} + \frac{\partial p'_{y'y'}}{\partial y'} \qquad \ldots(6)$$

$$\frac{\partial v'}{\partial y'} = 0 \qquad \ldots(7)$$

It is evident from (7) that v' is a function of time only. Hence, following Messiha, we consider v' as

$$v' = - v_0'(1 + \epsilon A \exp(in' t') \quad \text{...(8)}$$

where v_0' is a non-zero constant mean suction velocity, ϵ is small and A is a real positive constant such that $\epsilon A \leqslant 1$. The negative sign in (8) indicates that the suction velocity normal to the wall is directed towards the wall.

Equation (1) gives

$$p'_{x'x'} + \lambda_0 \left(\frac{\partial p'_{x'x'}}{\partial t'} + v' \frac{\partial p'_{x'x'}}{\partial y'} \right) = 0 \quad \text{...(9)}$$

$$p'_{x'y'} + \lambda_0 \left(\frac{\partial p'_{x'y'}}{\partial t'} + v' \frac{\partial p'_{x'y'}}{\partial y'} \right) = \mu \frac{\partial u'}{\partial y'} \quad \text{...(10)}$$

$$p'_{y'y'} + \lambda_0 \left(\frac{\partial p'_{y'y'}}{\partial t'} + v' \frac{\partial p'_{y'y'}}{\partial y'} \right) = 0. \quad \text{...(11)}$$

Clearly $p'_{x'x'} = 0$ and $p'_{y'y'} = 0$ are the particular solutions of eqns. (9) and (11).

Putting $p'_{y'y'} = 0$ in (6), we get

$$\rho' \frac{\partial v'}{\partial t'} = - \frac{\partial p'}{\partial y'}. \quad \text{...(12)}$$

Also from (8) and (12), as $\frac{\partial p'}{\partial y'}$ is small in the boundary layer, it can be neglected. Hence the pressure is taken to be constant along any normal and is given by its value outside the boundary layer. If $U'(t')$ is the free stream velocity, then eqn. (5) for the free stream becomes

$$- \frac{\partial p'}{\partial x'} = \rho' \frac{dU'}{dt'}. \quad \text{...(13)}$$

Equation (5) is then reduced to

$$\rho' \left(\frac{\partial u'}{\partial t'} + v' \frac{\partial u'}{\partial y'} \right) = \rho' \frac{dU'}{dt'} + \frac{\partial p'_{x'y'}}{\partial y'}. \quad \text{...(14)}$$

Eliminating $p'_{x'y'}$ in eqns. (10) and (14), we get

$$\frac{\partial u'}{\partial t'} + v' \frac{\partial u'}{\partial y'} + \lambda_0 \frac{\partial^2 u'}{\partial t'^2} + 2\lambda_0 v' \frac{\partial}{\partial y'} \left(\frac{\partial u'}{\partial t'} \right) + \lambda_0 \frac{\partial v'}{\partial t'} \frac{\partial u'}{\partial y'}$$
$$= \frac{dU'}{dt'} + \lambda_0 \frac{d^2 U'}{dt'^2} + (\nu - \lambda_0 v'^2) \frac{\partial^2 u'}{\partial y'^2}. \quad \text{...(15)}$$

The boundary conditions are

$$u' = L_1 \frac{\partial u'}{\partial y'} \text{ at } y' = 0 \text{ and } u' \to U'(t') \text{ as } y' \to \infty \quad \text{...(16)}$$

where $L_1 = \frac{2 - m_1}{m_1} L$; L being the mean free path and m_1 the Maxwell's reflexion coefficient.

Let us now consider a periodic free-stream velocity of the form

$$U'(t') = U'_0 (1 + \epsilon \exp (in' t')) \quad \ldots(17)$$

and let the velocity in the neighbourhood of the plate be

$$u'(y', t') = U'_0 [f_1(y') + \epsilon \exp (in' t') f_2(y')] \quad \ldots(18)$$

where n' is the frequency of the fluctuating stream, U'_0 the mean of $U'(t')$, $\epsilon U'_0$ the amplitude of the free stream fluctuation, $U'_0 f_1$ the mean velocity in the boundary layer and $\epsilon U'_0 f_2$ the amplitude of the velocity fluctuation in the boundary layer.

Substituting (8), (17) and (18) in (15), comparing harmonic terms and neglecting the coefficients of ϵ^2, we get

$$(\nu - \lambda_0 v_0'^2) f_1'' + v_0' f_1' = 0 \quad \ldots(19)$$

$$(\nu - \lambda_0 v_0'^2) f_2'' + (1 + 2\lambda_0 in') v_0' f_2' - (in' - \lambda_0 n'^2) f_2$$

$$= 2A\lambda_0 v_0'^2 f_1'' - (1 + \lambda_0 in') v_0' A f_1' - (in' - \lambda_0 n'^2) \quad \ldots(20)$$

where the primes denote differentiation with respect to y' and the non-dimensional quantities are defined as follows :

$$y = \frac{y' v_0'}{\nu},\ \lambda = \frac{\lambda_0 v_0'^2}{\nu},\ n = \frac{\nu n'}{v_0'^2},\ t = \frac{v_0'^2 t'}{\nu},\ U = \frac{U'}{U_0'} \text{ and } u = \frac{u'}{U_0'} \quad \ldots(21)$$

In view of (21), eqns. (19) and (20) now reduce to

$$(1 - \lambda) f_1'' + f_1' = 0 \quad \ldots(22)$$

$$(1 - \lambda) f_2'' + (1 + 2i\lambda n) f_2' - in(1 + i\lambda n) f_2 = 2A\lambda f_1''$$

$$- (1 + \lambda in) A f_1' - in(1 + \lambda in) \quad \ldots(23)$$

The boundary conditions are transformed to

$$\left.\begin{aligned} f_1 &= h_1 \frac{df_1}{dy},\ f_2 = h_1 \frac{df_2}{dy} \text{ at } y = 0 \\ f_1 &= f_2 = 1 \text{ as } y \to \infty \end{aligned}\right\} \quad \ldots(24)$$

where

$$h_1 = \frac{L_1 v_0'}{\nu}; \text{ the rarefaction parameter.}$$

The solutions of (22) and (23) satisfying (24) are

$$f_1 = 1 - \frac{(1-\lambda)e^{-y/(1-\lambda)}}{(1-\lambda+h_1)} \quad \text{...(25)}$$

$$f_2 = 1 + \frac{A\,e^{-y/(1-\lambda)}}{in(1-\lambda+h_1)} - \left[1 + \frac{A}{in(1-\lambda)}\right]\frac{e^{-hy}}{(1+hh_1)} \quad \text{...(26)}$$

where

$$h = \frac{(1+2in\lambda) + \sqrt{(1-4\lambda n^2)+4in}}{2(1-\lambda)}. \quad \text{...(27)}$$

It is noticed from (27) that $\lambda < 1$. Hence the velocity field in the boundary layer is given by

$$u(y,t) = 1 - \frac{(1-\lambda)\,e^{-y/(1-\lambda)}}{(1-\lambda+h_1)} + \epsilon\, e^{int}\left[1 + \frac{A\,e^{-y/(1-\lambda)}}{in(1-\lambda+h_1)} - \left\{1 + \frac{A}{in(1-\lambda)}\right\}\frac{e^{-hy}}{(1+hh_1)}\right]. \quad \text{...(28)}$$

The non-dimensional skin-friction τ_w is given by

$$\tau_w = (u)_{y=0} = h_1\left[\frac{1}{1-\lambda+h_1} + \epsilon\, e^{int}\left\{\frac{h}{1+hh_1} + \frac{A(h-1-\lambda h)}{in(1-\lambda)(1+hh_1)(1-\lambda+h_1)}\right\}\right]. \quad \text{...(29)}$$

It is observed from the expressions for the velocity and skin friction that their steady parts are also influenced by the elastic parameter λ and the rarefaction parameter h_1.

From (28), we get

$$u(y,t) = 1 - \frac{(1-\lambda)\,e^{-y/(1-\lambda)}}{(1-\lambda+h_1)} + \epsilon(M_{1r}\cos nt - M_{1i}\sin nt) \quad \text{...(30)}$$

where M_{1r} and M_{1i} are the fluctuating parts of the velocity profile and are given by

$$M_{1r} = M_r + \frac{AM_i}{n(1-\lambda)}$$

$$M_{1i} = M_i + \frac{A(1-M_r)}{n(1-\lambda)} - \frac{A\,e^{-y/(1-\lambda)}}{n(1-\lambda+h_1)}$$

$$M_r = 1 - \frac{\{(1+h_1h_r)\cos h_iy - h_1h_i\sin h_iy\}\,e^{-h_ry}}{(1+h_1h_r)^2 + h_1^2h_i^2}$$

$$M_i = \frac{\{(1+h_1h_r)\sin h_iy + h_1h_i\cos h_iy\}\,e^{-h_ry}}{(1+h_1h_r)^2 + h_1^2h_i^2}$$

$$h_{1r} = h_r + \frac{Ah_i}{n(1-\lambda+h_1)},\quad h_{1i} = h_i - \frac{A(h_r - 1 - \lambda h_r)}{n(1-\lambda)(1-\lambda+h_1)}$$

$$h_r = \frac{1 + \sqrt{\dfrac{R + (1 - 4\lambda n^2)}{2}}}{2(1-\lambda)},\quad h_i = \frac{2\lambda n + \sqrt{\dfrac{R - (1 - 4\lambda n^2)}{2}}}{2(1-\lambda)}$$

$$R = \sqrt{(1 - 4\lambda n^2)^2 + 16 n^2}\cdot$$

From eqn. (29), we get

$$\tau_w = h_1 \left[\frac{1}{(1-\lambda+h_1)} + \epsilon \mid B \mid \cos(nt+\alpha) \right] \qquad \ldots(31)$$

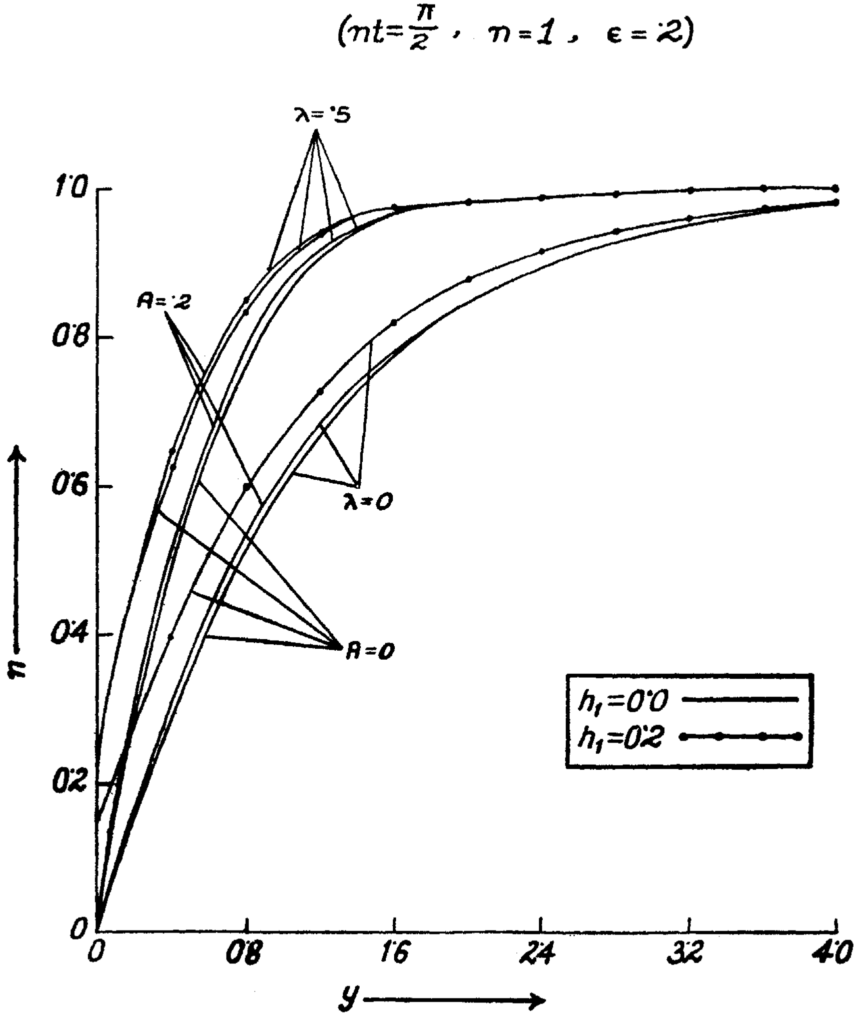

FIG. 1. Velocity profiles.

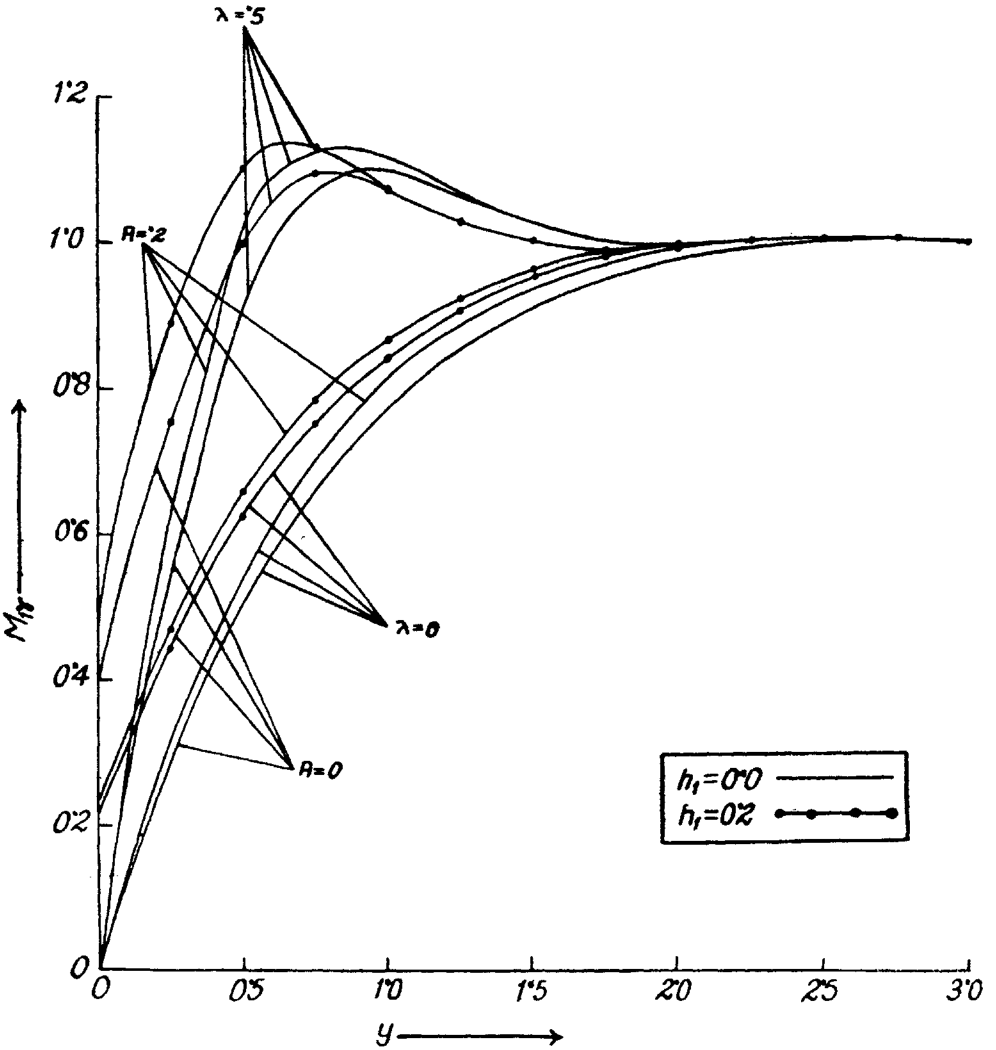

FIG. 2. Fluctuating part of velocity profiles.

where

$$|B| = \frac{[\{(1 + h_1 h_r)\, h_{1r} + h_1 h_i h_{1i}\}^2 + \{h_1 h_i h'_{1r} - (1 + h_1 h_r)\, h_{1i}\}^2]^{1/2}}{(1 + h_1 h_r)^2 + h_1^2\, h_i^2}$$

and

$$\tan \alpha = \frac{(1 + h_1 h_r)\, h_{1i} - h_1 h_i h_{1r}}{(1 + h_1 h_r)\, h_{1r} + h_1 h_i h_{1i}}.$$

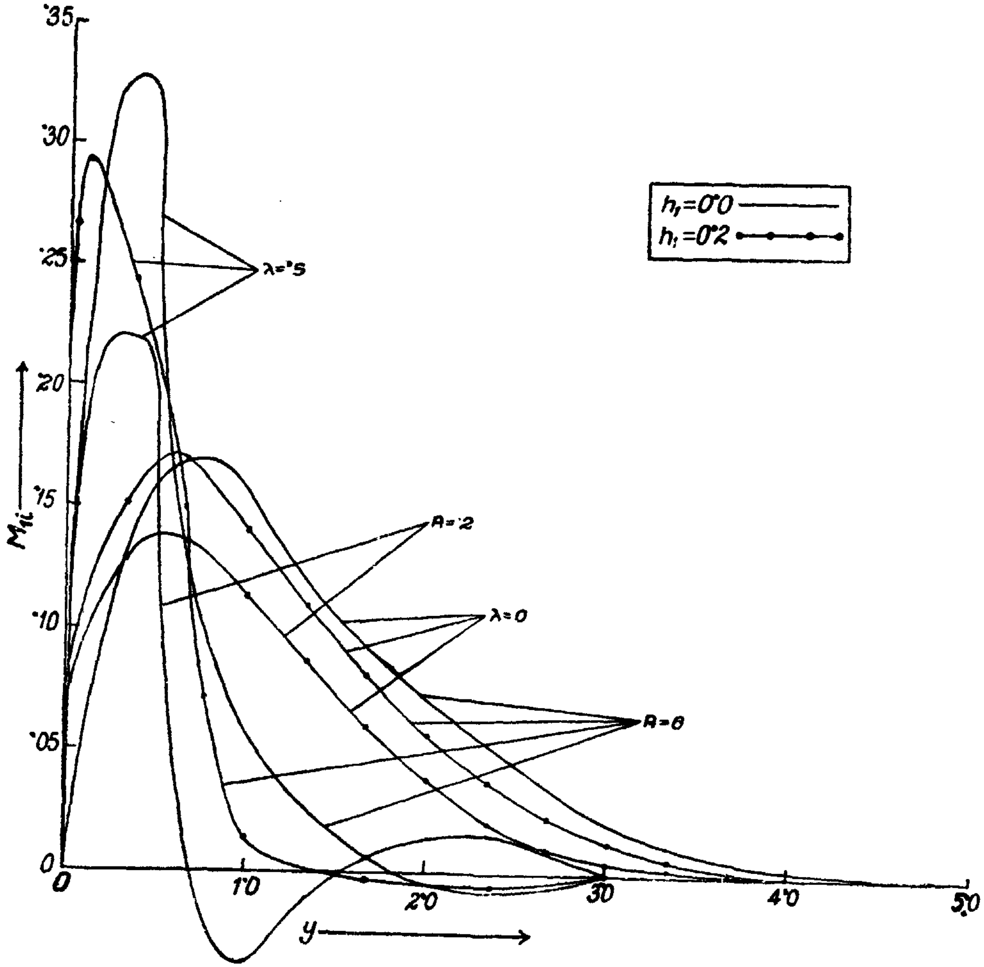

FIG. 3. Fluctuating part of velocity profiles.

PARTICULAR CASE : CONSTANT SUCTION VELOCITY

Substituting $A = 0$ in (30) and (31), we get

$$u(y, t) = 1 - \frac{(1 - \lambda)\, e^{-y/(1-\lambda)}}{(1 - \lambda + h_1)} + \epsilon(M_r \cos nt - M_i \sin nt) \qquad \ldots(32)$$

$$\tau_w = h_1 \left[\frac{1}{(1 - \lambda + h_1)} + \epsilon \mid B_0 \mid \cos(nt + \alpha_0) \right]$$

where

$$\mid B_0 \mid = \frac{[\{h_r + h_1(h_r^2 + h_i^2)\}^2 + h_i^2]^{1/2}}{(1 + h_1 h_r)^2 + h_1^2\, h_i^2}$$

and

$$\tan \alpha_0 = \frac{h_i}{h_r + h_1(h_r^2 + h_i^2)}.$$

DISCUSSION

In order to study the effects of the elasticity and rarefaction parameter on the distribution of the velocity profiles near the wall, both in case of constant and variable suction, we have plotted u against y in Fig. 1. It is clear from the figure that the velocity field in the boundary layer for viscoelastic fluids is more than that of Newtonian fluids for the same values of A and h_1. It is also clear that for constant values of λ and h_1, the velocity at any point of the fluid increases as A increases. Again we observe that an increase in h_1 increases the velocity for constant values of A and λ.

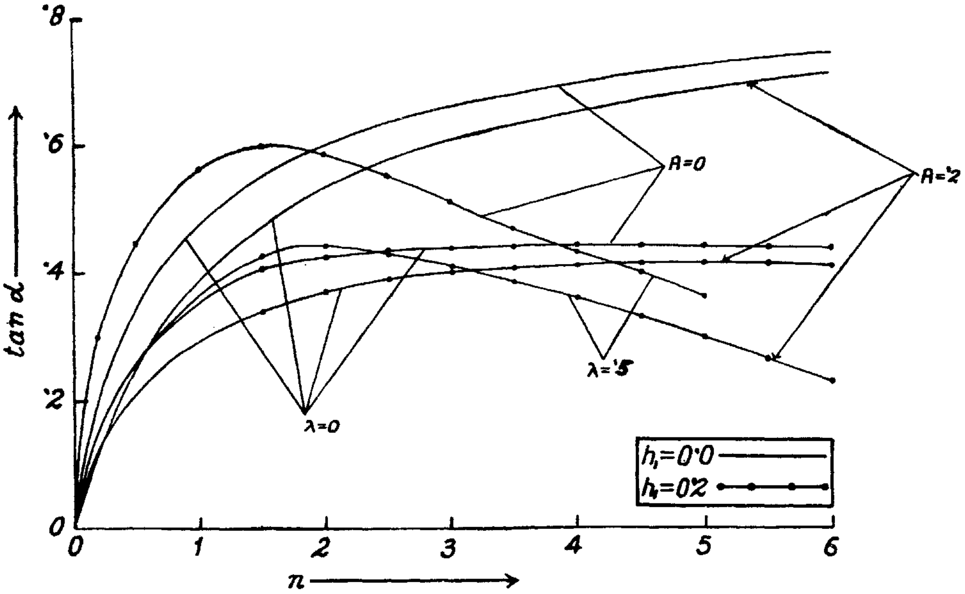

FIG. 4. Skin friction phase versus n.

TABLE I

Values of skin friction phase for $h_1 = 0$, $\lambda = .5$

n \ A	0.0	0.2
0	0.0	0.0
1	1.1559743	0.760121
2	2.1169676	1.466482
3	3.0874826	2.1720452
4	4.0686648	2.880311
5	5.056181	3.5904

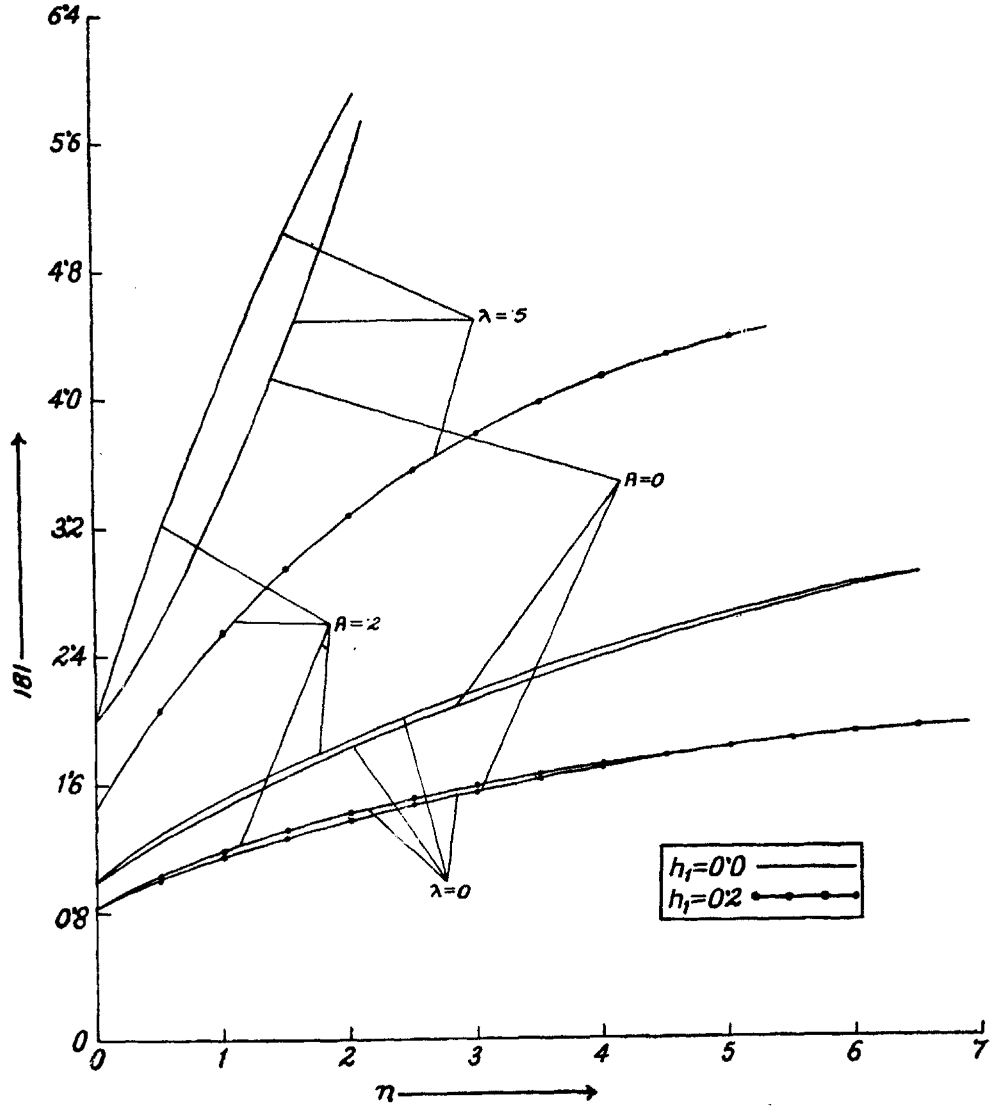

FIG. 5. Amplitude of skin friction against frequency n.

In Figs. 2 and 3, the details about the fluctuating parts are shown for $n = 1$. Figure 2 illustrates that M_{1r} for viscoelastic fluids is more than M_{1r} for Newtonian fluids. It is also obvious that M_{1r} increases with an increase in h_1. Also, from Figs. 2 and 3 one can conclude that an increase in A leads to an increase in M_{1r} but a decrease in M_{1i} for the same values of λ and h_1. Figure 3 is particularly interesting because there is a sudden rise and fall of M_{1i} near the wall, which is not observed in ordinary Newtonian fluids.

In Fig. 4, the skin-friction phase is shown against frequency. For the same value of λ, an increase in A or h_1 leads to a decrease in the phase of skin-friction. From Fig. 4, it is evident that at smaller frequency in slip-flow regime, the skin-friction phase increases when there is an increase in the elastic parameter λ while it decreases with an increase in the elastic parameter at large n in slip-flow regime. But from Fig. 4 and Table I, it is clear that in case of $h_1 = 0$, the skin-friction phase increases with the elastic parameter, both in case of smaller and larger values of n.

Figure 5 illustrates the effects of λ, h_1 and A on the amplitude of the skin-friction. An increase in λ or A increases the amplitude of the skin-friction for the same value of h_1. But for the same values of λ and A, an increase in h_1 leads to a decrease in $|B|$.

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