ON THE ABSOLUTE EULER SUMMABILITY FACTORS FOR FOURIER SERIES AND ITS CONJUGATE SERIES

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(*Received* 1 *December* 1976; *after revision* 28 *June* 1978)

In this paper, the author has obtained a sequence of factors $\{(\log (n + 1))^{-1}\}$ to ensure absolute Euler summability of Fourier series and its conjugate series of the class of functions of bounded variation. He has further investigated that, for Fourier series, the above class of functions may be replaced by another class of functions which is not strictly contained in the former class.

1. Definitions and Notations

Let $\sum_{n=0}^{\infty} d_n$ be a given infinite series. Then, for $p > -1$, we write

$$d_n^p = (1 + p)^{-n-1} \sum_{m=0}^{n} \binom{n}{m} p^{n-m} d_m; \; d_n^0 = d_n, \qquad \ldots(1.1)$$

where $\sum_{n=0}^{\infty} d_n^p$ is called pth Euler transform of $\sum_{n=0}^{\infty} d_n$ and we write

$$\sum_{n=0}^{\infty} d_n \in |E, p| \Leftrightarrow \sum_{n=1}^{\infty} |d_n^p| < \infty. \qquad \ldots(1.2)$$

Earlier Lorentz (1948, p. 180) defined Euler method of summation $E_\alpha (\alpha > 0)$ and later Knopp and Lorentz (1949, p. 13) defined E_k and $|E_k|$ methods of summation for $k \geqslant 1$. These notations are different from (E, p) $(p \geqslant 0)$, which is given in Hardy (1949), on p. 180. However, it may be observed that, for $p = k - 1$ $(k \geqslant 1)$, $(E, p) = E_k$. For $|E, p|$ $(p \geqslant 0)$, a reference may be made to Hardy (1949, p. 237). Recently the present author (Chandra 1975)* has, independently, extended the definitions of (E, p) and $|E, p|$ summability methods for all real or complex p except $p \neq -1$.

Let f be 2π-periodic and L-integrable over $(-\pi, \pi)$, and let its Fourier series, at a point x, be

*Also see *Mathematical Reviews*, 54 (1977), 119-120.

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x). \qquad \ldots(1.3)$$

Then the series, conjugate to (1.3), will be given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \qquad \ldots(1.4)$$

Throughout the paper we assume $a_0 = 0$ and $B_0(x) = 0$. We, now, write for fixed real number x

$$\phi(t) = \tfrac{1}{2}\{f(x + t) + f(x - t)\} \qquad \ldots(1.5)$$

$$\psi(t) = \tfrac{1}{2}\{f(x + t) - f(x - t)\} \qquad \ldots(1.6)$$

$$P(t) = \phi(t) - \frac{1}{t} \int_0^t \phi(u)\, du \qquad \ldots(1.7)$$

$$g(t) = \phi(t)/\log \frac{k}{t} \quad (k > \pi e) \qquad \ldots(1.8)$$

$$v_m^p(n) = (1 + p)^{-n-1} \binom{n+1}{m+1} p^{n-m} \quad (m \geqslant 0,\ p > 0) \qquad \ldots(1.9)$$

$$y_n = (\log (n + 1))^{-1} \qquad \ldots(1.10)$$

$$\log_1 = \log,\ \log_b = \log \log_{b-1}, \qquad \ldots(1.11)$$

(where b is a positive integer).

2. Introduction

Several authors, such as Mohanty and Mohapatra (1968), Chandra (1972a, 1972b, 1973a, 1977), have obtained sufficient conditions, imposed upon the generating functions, to ensure $|E, p|$ $(p > 0)$ summability of Fourier series and its conjugate series. Recently, Tripathy (1969) has shown that $\phi(t) \in BV(0, \pi)$ does not ensure $|E, p|$ $(p > 0)$ summability of Fourier series at a point $t = x$.

One of the objects of this paper is to obtain a sequence of factors $\{y_n\}$ for $|E, p|$ $(p > 0)$ summability of Fourier series and its conjugate series of functions of bounded variation. Consequently we also show, in Section 5.4, that the same sequence of factors also ensures $|E, p|$ $(p > 0)$ summability of Fourier series of a function which is not of bounded variation. Precisely we prove the following theorems :

Theorem 1 — $\phi(t) \in BV(0, \pi)$ $\qquad \ldots(2.1)$

is a sufficient condition for

$$\sum_{n=1}^{\infty} A_n(x)\, y_n \in |\, E, p \,|\ (p > 0). \qquad \ldots(2.2)$$

However,

$$\int_0^{\pi} \log \frac{k}{t}\, |\, dg(t) \,| < \infty \quad (k > \pi e) \qquad \ldots(2.3)$$

is not a sufficient condition for (2.2).

Theorem 2 — Let $c > 1$ and $k > \pi \exp(\exp(c))$. Then

$$P(t)\left(\log_2 \frac{k}{t}\right)^c \in BV(0, \pi) \qquad \ldots(2.4)$$

is a sufficient condition for (2.2). However,

$$\int_0^{\pi} \log \frac{k}{t}\, |\, dP(t) \,| < \infty \qquad \ldots(2.5)$$

is not a sufficient condition for (2.2).

Theorem 3 — Condition

$$\text{(i)}\quad \psi(+0) = 0, \quad \text{(ii)}\quad \psi(t) \in BV(0, \pi) \qquad \ldots(2.6)$$

is sufficient for

$$\sum_{n=1}^{\infty} B_n(x)\, y_n \in |\, E, p \,|\ (p > 0). \qquad \ldots(2.7)$$

Theorem 4 — Conditions (2.6) (ii) and

$$t^{-1}\psi(t) \in L(0, \pi) \qquad \ldots(2.8)$$

are sufficient for (2.7).

3. Lemmas

We shall use the following lemmas in the proof of the theorems :

Lemma 1 — Let $p > q > -1$. Then $\sum_{n=0}^{\infty} a_n \in |\, E, q \,| \Rightarrow \sum_{n=0}^{\infty} a_n \in |\, E, p \,|$ (*see* Chandra 1975, Corollary 2).

Lemma 2 — Let $p > 0$. Then, uniformly in $0 < t \leqslant \pi$,

$$\sum_{m=1}^{n} y_m v_m^p(n) \exp(imt) = O\left\{t^{-1}\left(\log \frac{2\pi}{t}\right)^{-1} n^{-1/2}\right\}.$$

PROOF : Let $s = \left[\frac{n+1-p}{1+p}\right]$, that is the integral part of $\frac{(n+1-p)}{(1+p)}$. Then it may be observed that

$$v_{m-1}^p(n) < v_m^p(n) \text{ for } 0 \leqslant m \leqslant s, \text{ if } \frac{n+1-p}{1+p} \text{ is not an integer;} \qquad \ldots(3.1)$$

$$v_{m-1}^p(n) = v_m^p(n) \text{ for } m = s, \text{ if } \frac{n+1-p}{1+p} \text{ is an integer; and} \qquad \ldots(3.2)$$

$$v_{m-1}^p(n) > v_m^p(n) \text{ for } n \geqslant m > s. \qquad \ldots(3.3)$$

And hence it follows that $v_s^p(n)$ is maximum. We write

$$\sum_{m=1}^{n} v_m^p(n)\, y_m \exp(imt) = \left(\sum_{m=1}^{s} + \sum_{s+1}^{n}\right)(v_m^p(n)\, y_m \exp(imt)$$
$$= \Sigma_1 + \Sigma_2, \text{ say.}$$

Since $\{v_m^p(n)\}$ is positive and monotonic increasing for $0 \leqslant m \leqslant s$, we obtain by Abel's lemma that

$$\left|\sum\nolimits_1\right| \leqslant v_s^p(n) \max_{1 \leqslant m' \leqslant s} \left|\sum_{m=m'}^{s} y_m \exp(imt)\right|$$
$$= O\left\{t^{-1}\left(\log \frac{2\pi}{t}\right)^{-1} v_s^p(n)\right\},$$

since, in the case $s > T = \left[\frac{2\pi}{t}\right] > m'$,

$$\left|\sum_{m=m'}^{s} y_m \exp(imt)\right| \leqslant \left|\sum_{m=m'}^{T} y_m \exp(imt)\right| + \left|\sum_{m=1+T}^{s} y_m \exp(imt)\right|$$
$$= O\left\{\sum_{m=m'}^{T} y_m\right\} + O\left\{\left(\log \frac{2\pi}{t}\right)^{-1} \times\right.$$

(*equation continued on p.* 1008)

$$\times \max_{1+T\leqslant s'\leqslant s}\left|\sum_{m=1+T}^{s'} \exp(imt)\right|\Bigg\}$$

$$= O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^{-1}\right\},$$

uniformly in $0 < t \leqslant \pi$, and, whenever $s \leqslant T$, we obtain that

$$\left|\sum_{m=m'}^{s} y_m \exp(imt)\right| \leqslant \sum_{m=m'}^{T} y_m = O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^{-1}\right\},$$

uniformly in $0 < t \leqslant \pi$. Finally, in the case $T \leqslant m'$, we have

$$\left|\sum_{m=m'}^{s} y_m \exp(imt)\right| \leqslant \left|\sum_{m=T}^{s} y_m \exp(imt)\right| + \left|\sum_{m=T}^{m'-1} y_m \exp(imt)\right|$$

$$= O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^{-1}\right\},$$

uniformly in $0< t \leqslant \pi$.

Also, we observe that $\{v_m^p(n)\}$ is positive and monotonic decreasing for $s<m\leqslant n$, and thus by using Abel's lemma we obtain that

$$\sum\nolimits_2 \leqslant v_s^p(n) \max_{s+1\leqslant n'\leqslant n}\left|\sum_{m=s+1}^{n'} y_m \exp(imt)\right|$$

$$= O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^{-1} v_s^p(n)\right\},$$

uniformly in $0 < t \leqslant \pi$, proceeding as in Σ_1.

Combining the estimates, obtained above, for Σ_1 and Σ_2, we obtain that

$$\sum_{m=1}^{n} y_m v_m^p(n) \exp(imt) = O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^{-1} v_s^p(n)\right\}, \qquad \ldots(3.4)$$

uniformly in $0 < t \leqslant \pi$, where $s = \left[\frac{n+1-p}{1+p}\right]$.

Finally, by using Stirling's asymptotic values (*see* Hobson 1957, p. 70; also Bromwich 1959, p. 509) for

$\Gamma(n+1)$, $\Gamma(m+1)$ and $\Gamma(n-m+1)$, it may be obtained that

$$v_s^p(n) = O(n^{-1/2})$$

and substituting this estimate in (3.4), the proof of the lemma follows.

4. Proof of Theorem 1

(A) (2.1) *is sufficient for* (2.2) — We have

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \phi(t) \cos nt \, dt.$$

Integrating by parts, we obtain that

$$A_n(x) = -\frac{2}{\pi} \int_0^\pi \frac{\sin nt}{n} \, d\phi(t) \quad (n > 0).$$

In view of (1.1), (1.2) and the fact that $A_0(x) = 0$, it will be enough for the proof of (A) to show that

$$\sum = \sum_{n=1}^{\infty} \left| \sum_{m=1}^{n} v_{m-1}^p (n-1) A_m(x) y_m \right| < \infty.$$

But

$$\sum \leqslant \frac{2}{\pi} \int_0^\pi | d\phi(t) | \sum_{n=1}^{\infty} \left| \sum_{m=1}^{n} v_{m-1}^p (n-1) \frac{\sin mt}{m+1} \right| + O(1).$$

Now, by (2.1), it is enough to show that

$$S = \sum_{n=1}^{\infty} \frac{1}{n+1} \left| \sum_{m=1}^{n} y_m v_m^p (n) \sin nt \right| = O(1),$$

uniformly in $0 < t \leqslant \pi$.

For $T_1 = \left[\frac{2\pi}{t}\right]$ and $T_2 = \left[\left(\frac{2\pi}{t}\right)^2\right]$, we write

$$S = \sum_{n=1}^{T_1} + \sum_{1+T_1}^{T_2} + \sum_{1+T_2}^{\infty} = \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say.}$$

Now, it may be observed that the inner sum of Σ_1 does not exceed

$$O\{t(1-p/(1+p))^{-(m+1)}\}$$

and therefore $\Sigma_1 = O(1)$, uniformly in $0 < t \leqslant \pi$, and, by using twice the relation

$$\binom{a+1}{b+1} = \frac{a+1}{b+1}\binom{a}{b},$$

it may be seen that the inner sum of Σ_2 does not exceed

$$\frac{y_n}{n+1}\sum_{m=0}^{n+2}\binom{n+2}{m}p^{n+2-m}$$

and hence uniform boundedness, in $0 < t \leqslant \pi$, of Σ_2 immediately follows. Finally, by Lemma 2,

$$\sum\nolimits_3 = O\left\{\left(t\log\frac{2\pi}{t}\right)^{-1}\sum_{n=T_2}^{\infty} n^{-3/2}\right\}$$

$$= O(1),$$

uniformly in $0 < t \leqslant \pi$. This completes the proof of (A).

(B) (2.3) *is not a sufficient condition for* (2.2) — We have

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} g(t)\log\frac{k}{t}\cos nt\,dt$$

$$= \frac{2}{\pi}\left[-g(t)\int_t^{\pi}\log\frac{k}{u}\cos nu\,du\right]_0^{\pi}$$

$$+\frac{2}{\pi}\int_0^{\pi} dg(t)\int_t^{\pi}\log\frac{k}{u}\cos nu\,du$$

$$= \frac{2}{\pi}g(+0)\int_0^{\pi}\log\frac{k}{u}\cos nu\,du$$

$$+\frac{2}{\pi}\int_0^{\pi}\log\frac{k}{t}\,dg(t)\int_t^{\theta}\cos nu\,du \quad (t \leqslant \theta \leqslant \pi)$$

$$= \frac{2}{\pi} g(+0) \int_0^\pi \frac{\sin nt}{nt} \, dt$$

$$+ \frac{2}{n\pi} \int_0^\pi \log \frac{k}{t} \, dg(t) \, (\sin n\theta - \sin nt)$$

$$= \frac{2}{\pi} g(+0) \left\{ \int_0^\pi \frac{\sin nt}{nt} \, dt - \frac{\pi}{2n} \right\} + \frac{g(+0)}{n}$$

$$+ \frac{2}{n\pi} \int_0^\pi \log \frac{k}{t} \, dg(t) \, (\sin n\theta - \sin nt)$$

$$= \alpha_n + \beta_n + \gamma_n, \text{ say.}$$

Thus

$$\beta_n = A_n(x) - \alpha_n - \gamma_n. \qquad \ldots(4.1)$$

Proceeding as in (A) of this theorem, we can show that

$$\sum_{n=1}^{\infty} \gamma_n y_n \in | E, p | \; (p > 0),$$

whenever (2.3) holds, and, since

$$\int_0^\pi \frac{\sin nt}{nt} \, dt = \frac{1}{n} \int_0^{n\pi} \frac{\sin t}{t} \, dt$$

$$= \frac{1}{n} \left[\int_0^\infty \frac{\sin t}{t} \, dt - \int_{n\pi}^\infty \frac{\sin t}{t} \, dt \right]$$

$$= \frac{\pi}{2n} + O(n^{-2}),$$

$$\alpha_n = O(n^{-2}). \qquad \ldots(4.2)$$

Therefore, by Lemma 1, $\sum_{n=1}^{\infty} y_n \alpha_n \in | E, p | \; (p > 0)$. Now, in order that (2.2) should hold, it is necessary that

$$\sum_{n=1}^{\infty} y_n \beta_n \in | E, p | \; (p > 0),$$

for which it is necessary and sufficient that

$$\Sigma = \sum_{n=1}^{\infty} \sum_{m=1}^{n} m^{-1} y_m v_{m-1}^{p} (n-1) < \infty. \qquad \ldots(4.3)$$

However, by changing the order of summation, it follows that

$$\Sigma = \infty. \qquad \ldots(4.4)$$

Thus (2.2) does not hold and this completes the proof of Theorem 1.

5. Proof of Theorem 2

(A) (2.4) *is sufficient* — Proceeding as in Theorem 1 of Chandra (1973c), we obtain that

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} P(t) \cos nt \, dt - \frac{2}{\pi} \int_0^{\pi} P(t) \frac{\sin nt}{nt} \, dt$$

$$= A_n^{(1)}(x) - A_n^{(2)}(x), \text{ say.} \qquad \ldots(5.1)$$

For the proof of (A), it is sufficient to show that each of the series $\sum_{n=1}^{\infty} A_n^{(1)}(x) y_n$ and $\sum_{n=1}^{\infty} A_n^{(2)}(x) y_n$ is summable $|E, p|$ $(p > 0)$. However, from (A) of Theorem 1 it follows that the former series is summable $|E, p|$ $(p > 0)$, since

$$P(t)\left(\log_2 \frac{k}{t}\right)^c \in BV(0, \pi) \Rightarrow P(t) \in BV(0, \pi). \text{ And}$$

$$P(t)\left(\log_2 \frac{k}{t}\right)^c \in BV(0, \pi) \Leftrightarrow P(t)\left(\log_2 \frac{k}{t}\right)^c = F_1(t) - F_2(t),$$

where $F_i(t)$ $(i = 1, 2)$ are positive, bounded and monotonic increasing in $(0, \pi)$. Therefore, by using the second mean value theorem, we obtain that

$$A_n^{(2)}(x) = \frac{2}{n\pi} \{F_1(\pi) - F_2(\pi)\} \int_0^{\pi} \frac{\sin nt}{t\left(\log_2 \frac{k}{t}\right)^c} \, dt$$

$$+ F_2(\pi) \int_0^{\eta} \frac{\sin nt}{t\left(\log_2 \frac{k}{t}\right)^c} \, dt - F_1(\pi) \int_0^{\eta'} \frac{\sin nt}{t\left(\log_2 \frac{k}{t}\right)^c} \, dt$$

$$(0 \leqslant \eta \leqslant \pi) \qquad\qquad (0 \leqslant \eta' \leqslant \pi)$$

$$= O\{(n+1)^{-1} (\log_2(n+3))^{-c}\},$$

since, by (3.2) of Chandra (1973b),

$$\int_0^t \frac{\sin nu}{u\left(\log_2 \frac{k}{u}\right)^c} du = O\{(\log_2(n+3))^{-c}\},$$

uniformly in $0 < t \leqslant \pi$. Therefore, it follows by Lemma 1, that the later series is also summable $| E, p | \ (p > 0)$.

(B) (2.5) *is not sufficient* — From (5.1),

$$A_n(x) = A_n^{(1)}(x) - A_n^{(2)}(x),$$

where

$$A_n^{(2)}(x) = \frac{2}{\pi} \int_0^\pi P(t) \frac{\sin nt}{nt} dt$$

$$= \frac{2}{\pi} P(+0) \int_0^\pi \frac{\sin nt}{nt} dt + \frac{2}{\pi} \int_0^\pi dP(t) \int_t^\pi \frac{\sin nu}{nu} du$$

$$= \frac{2}{\pi} P(+0) \left\{ \int_0^\pi \frac{\sin nt}{nt} dt - \frac{\pi}{2n} \right\} + n^{-1} P(+0)$$

$$+ \frac{2}{\pi} \int_0^\pi \log \frac{k}{t} \, dP(t) \frac{\sin n\theta}{n} \quad (t \leqslant \theta \leqslant \pi)$$

$$= \alpha_n' + \beta_n' + \gamma_n', \text{ say.}$$

By using (4.1) and Lemma 1, it follows that

$$\sum_{n=1}^{\infty} \alpha_n' y_n \in | E, p | \ (p > 0)$$

and proceeding as in (A) of Theorem 1, we may follow that each of the series $\sum_{n=1}^{\infty} \gamma_n' y_n$ and $\sum_{n=1}^{\infty} A_n^{(1)}(x) y_n$ is summable $| E, p | \ (p > 0)$. Thus, in order that (2.2) should hold, it is necessary that $\sum_{n=1}^{\infty} \beta'_n y_n \in |E, p| \ (p > 0)$ for which it is necessary that (4.3) should hold. But from (4.4), it follows that (4.3) is not true and hence (2.5) is not a sufficient condition for (2.2).

This completes the proof of Theorem 2.

6. Remarks

In this section we shall give the following remarks concerning Theorems 1 and 2.

5.1. Let $g(+0) = 0$ in Theorem 1. Then α_n and β_n both vanish and therefore we get the following :

Theorem 1′ — $g(+0) = 0$ and (2.3) are sufficient for (2.2).

Similarly, on the assumption $P(+0) = 0$, we obtain the following :

Theorem 2′ — $P(+0) = 0$ and (2.5) are sufficient for (2.2).

It may be observed that, in general, (2.5) does not imply (2.3).

5.2. It may be observed (*see* Chandra 1978, Corollary 2) that $\phi(t) \in BV(0, \pi)$ if and only if $P(t) \in BV(0, \pi)$ and $t^{-1}P(t) \in L(0, \pi)$. Therefore, (2.1) and (2.4) are not equivalent.

5.3. In (2.4), the factor $\left(\log_2 \frac{k}{t}\right)^c$ of $P(t)$ may be replaced by any one of the following :

$$\log_2 \frac{k}{t} \left(\log_3 \frac{k}{t}\right)^c, \ldots, \log_2 \frac{k}{t} \ldots \log_{b-1} \frac{k}{t} \left(\log_b \frac{k}{t}\right)^c,$$

where $c > 1$ and k is a suitable positive constant taken for the convenience in the analysis and not necessarily the same in each of the functions defined above.

5.4. Let 2π-periodic function f be even so that, for $x = 0$, $\phi(t) = f(t)$. And let

$$f(t) = \left(\log \frac{k}{t}\right)^h (0 < t \leqslant \pi,\ 0 < h < 1),$$

where $k \geqslant \pi \exp\left\{\exp\left(\frac{2c}{(1-h)}\right)\right\}$ for $c > 1$. Then, for $\phi(t) = \left(\log \frac{k}{t}\right)^h$, it may be obtained that

$$P(t)\left(\log_2 \frac{k}{t}\right)^c = -h\left(\log \frac{k}{t}\right)^{h-1}\left(\log_2 \frac{k}{t}\right)^c$$

$$+ h(1-h)\, t^{-1}\left(\log_2 \frac{k}{t}\right)^c \int_0^t \left(\log \frac{k}{u}\right)^{h-2} du$$

$$= -hX(t) + h(1-h)\, Y(t),$$

where

$$\int_0^{\pi} | dX(t) | = O(1)$$

and

$$\int_0^{\pi} | dY(t) | \leqslant 2 \int_0^{\pi} t^{-1}\left(\log_2 \frac{k}{t}\right)^c \left(\log \frac{k}{t}\right)^{h-2} dt$$

$$+ \frac{1}{\pi}\left(\log_2 \frac{k}{\pi}\right)^c \int_0^{\pi} \left(\log \frac{k}{t}\right)^{h-2} dt$$

$$= O(1).$$

Hence it follows that $P(t)\left(\log_2 \frac{k}{t}\right)^c \in BV(0, \pi)$. Now, we follow from (A) of Theorem 2 that the factor y_n is also a summability factor for $| E, p |$ summability of a Fourier series of the function $\left(\log \frac{k}{t}\right)^h$, defined above, which is not of bounded variation over $(0, \pi)$.

7. Proof of Theorem 3

We have

$$B_n(x) = \frac{2}{\pi} \int_0^{\pi} \psi(t) \sin nt \, dt.$$

Integrating by parts and using the fact that $\psi(\pi) = 0$, we obtain that

$$B_n(x) = \frac{2}{n\pi} \int_0^{\pi} \cos nt \, d\psi(t). \qquad ...(6.1)$$

By (2.6i) and the fact that $\psi(\pi) = 0$,

$$\int_0^{\pi} d\psi(t) = 0$$

and hence we obtain that

$$B_n(x) = -\frac{4}{n\pi} \int_0^{\pi} \sin^2 \tfrac{1}{2} nt \, d\psi(t). \qquad ...(6.2)$$

Now, for the proof of Theorem 3, it is enough to show that

$$\Sigma = \sum_{n=1}^{\infty} \left| \sum_{m=1}^{n} v_{m-1}^{p} (n-1)\, y_m B_m(x) \right| < \infty. \qquad \ldots(6.3)$$

For T_1 as defined in Theorem 1, we write

$$\Sigma = \sum_{n=1}^{T_1} + \sum_{1+T_1}^{\infty} = \Sigma_1 + \Sigma_2, \text{ say.}$$

By using (6.2) and proceeding as in Σ_1 of Theorem 1, we obtain that

$$\begin{aligned}\sum\nolimits_1 &\leqslant \frac{4}{\pi} \int_0^{\pi} |\, d\psi(t)\,| \sum_{n=1}^{T_1} \sum_{m=1}^{n} v_{m-1}^{p} (n-1)\, y_m\, m^{-1} \sin^2 \tfrac{1}{2} mt \\ &\leqslant \frac{2}{\pi} \int_0^{\pi} t\, |\, d\psi(t)\,| \sum_{n=1}^{T_1} \sum_{m=1}^{n} v_{m-1}^{p} (n-1)\, y_m \\ &< \infty,\end{aligned}$$

by (2.6 ii). And, by (6.1),

$$\sum\nolimits_2 \leqslant \frac{2}{\pi} \int_0^{\pi} |\, d\psi(t)\,| \sum_{n=T_1}^{\infty} \left| \sum_{m=1}^{n} v_{m-1}^{p} (n-1)\, y_m (m+1)^{-1} \cos mt \right| + O(1).$$

Now, for T_2 as defined in Theorem 1, we further write

$$\begin{aligned}&\sum_{n=T_1}^{\infty} \left| \Sigma\, v_{m-1}^{p} (n-1)\, y_m (m+1)^{-1} \cos mt \right| \\ &= \sum_{n=T_1}^{T_2} + \sum_{1+T_2}^{\infty} = \Sigma_{2,1} + \Sigma_{2,2}, \text{ say.}\end{aligned}$$

Proceeding as in Σ_i $(i = 2, 3)$ of Theorem 1, we obtain that

$$\Sigma_{2,j} = O(1) \quad (j = 1, 2),$$

uniformly in $0 < t \leqslant \pi$. And thus, by (2.6ii), we follow that

$$\Sigma_2 = O(1).$$

And this completes the proof of Theorem 3.

8. Proof of Theorem 4

We have

$$B_n(x) = \frac{2}{\pi} \int_0^\pi t\psi(t) \frac{\sin nt}{t}\, dt$$

$$= \frac{2}{\pi} \Big[- t\psi(t) \int_t^\pi \frac{\sin nu}{u}\, du \Big]_0^\pi$$

$$+ \frac{2}{\pi} \int_0^\pi d\{t\psi(t)\} \int_t^\pi \frac{\sin nu}{u}\, du$$

$$= \frac{2}{\pi} \int_0^\pi d\{t\psi(t)\} \int_t^\pi \frac{\sin nu}{u}\, du.$$

Thus $\sum_{n=1}^{\infty} B_n(x)\, y_n \in |\, E, p\, |\ (p > 0)$, if (6.3) holds, for which it will be sufficient to show that

$$\sum_{n=1}^{\infty} \Big| \sum_{m=1}^{n} y_m\, v_{m-1}^{p}\, (n-1) \int_t^\pi \frac{\sin mu}{u}\, du \Big| = O(t^{-1}),$$

uniformly in $0 < t < \pi$, since, by (2.6ii) and (2.8)

$$\int_0^\pi \frac{|\, d(t\psi(t)\, |}{t} \leqslant \int_0^\pi |\, d\psi(t)\, | + \int_0^\pi \frac{|\, \psi(t)\, |}{t}\, dt < \infty.$$

Now, breaking up the summation $\sum_{n=1}^{\infty}$ into three summations

$\Sigma_1 = \sum_{1}^{T_1}$, $\Sigma_2 = \sum_{1+T_1}^{T_2}$ and $\Sigma_3 = \sum_{1+T_2}^{\infty}$, where T_1 and T_2 are defined in Theorem 1, and using

$$\int_t^\pi \frac{\sin nu}{u}\, du = \begin{cases} O(1) \\ O((nt)^{-1}) \\ \qquad \dfrac{\cos nt - \cos n\theta}{nt}\ (t \leqslant \theta \leqslant \pi) \end{cases}$$

respectively in Σ_1, Σ_2 and Σ_3 and proceeding as in Theorem 3, the proof of Theorem 4 may be obtained.

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