RADIATING DEMIANSKI-TYPE SPACE-TIMES

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A generalized Demianski-type metric is considered in connection with null radiation fields. A new solution of the field equations $R_{ik} = -8\pi\sigma\xi_i\xi_k$, $\xi_i\xi^i = 0$ is obtained by the method of exterior-calculus of differential forms. Demianski empty space-time is derived as a particular case.

1. INTRODUCTION

The two empty space-time solutions of Einstein's equations admitting twisting, geodetic and shear-free null rays, which are well-known in the literature, are Kerr solution (Kerr 1963) and the so-called NUT solution (Newman *et al.* 1963). The Kerr metric can be expressed as (Debney *et al.* 1969, Hogan 1975)

$$\begin{aligned} ds^2 &= 2(du - k\sin^2\alpha\, d\beta)(dr + k\sin^2\alpha\, d\beta) \\ &+ \left(1 - \frac{2mr}{r^2 + k^2\cos^2\alpha}\right)(du - k\sin^2\alpha\, d\beta)^2 \\ &- (r^2 + k^2\cos^2\alpha)(d\alpha^2 + \sin^2\alpha\, d\beta^2) \qquad \ldots(1.1) \end{aligned}$$

where m and k are constants of integration. These constants m and k are interpreted as mass and angular momentum per unit mass respectively. The NUT metric can be given by

$$\begin{aligned} ds^2 &= 2(du + 2b\cos\alpha\, d\beta)\, dr - (r^2 + b^2)(d\alpha^2 + \sin^2\alpha\, d\beta^2) \\ &+ \left(1 - \frac{2mr + 2b^2}{r^2 + b^2}\right)(du + 2b\cos\alpha\, d\beta)^2 \qquad \ldots(1.2) \end{aligned}$$

where the constants m and b represent respectively the mass and magnetic monopole-type mass.

Using the method of complex co-ordinate transformation, a new solution of Einstein's vacuum field equations, depending on four arbitrary constants, is obtained by Demianski (1972). The geometry of this solution is described by the line element

$$\begin{aligned} ds^2 &= 2(du - W\, d\beta)\,[dr + (W + 2b\cos\alpha)\, d\beta] \\ &+ \left(1 - \frac{2mr + 2bF}{r^2 + F^2}\right)(du - W\, d\beta)^2 \\ &- (r^2 + F^2)(d\alpha^2 + \sin^2\alpha\, d\beta^2) \qquad \ldots(1.3) \end{aligned}$$

with $F = k \cos\alpha + c \cos\alpha \log\tan\frac{\alpha}{2} + c + b$,

$$W = k \sin^2\alpha - 2b\cos\alpha + c\sin^2\alpha \log\tan\frac{\alpha}{2} - c\cos\alpha. \qquad \ldots(1.4)$$

Here k, m, b and c are arbitrary constants. It is evident from (1.3) and (1.4) that the choice $k = c = 0$ gives us the NUT metric (1.2) and the choice $c = b = 0$ leads us to the Kerr metric (1.1). Thus Kerr and NUT solutions are particular cases of Demianski's solution.

It will be seen that if one consider the line-element

$$\begin{aligned} ds^2 &= 2(du + G\sin\alpha\, d\beta)(dr + H\sin\alpha\, d\beta) \\ &\quad - 2L(du + G\sin\alpha\, d\beta)^2 - (X^2 + Y^2)(d\alpha^2 + \sin^2\alpha\, d\beta^2) \qquad \ldots(1.5) \end{aligned}$$

with $G = G(\alpha)$, $H = H(\alpha)$, $X = X(u, r)$, $Y = Y(\alpha)$ and $L = L(u, \alpha, r)$ one can get the metric (1.3) out of it by suitable choice of the functions G, H, X, Y, L. We designate the metric (1.5) as the Demianski-type metric.

Vaidya *et al.* (1976) have investigated the combined Kerr-NUT metric in detail. They have also derived several exact solutions of Einstein's equations describing empty or radiating space-times. The object of the present investigation is to discuss radiating space-time manifolds with Demianski-type metrics.

2. Equations of Structure

Physics as well as geometry look very simple when one tries to see it locally and this was the motivation behind Cartan's approach of exterior differential forms. In this approach the problems of Riemannian geometry are reduced to the problems of Euclidean geometry. We shall use this approach in the present paper. This technique of differential forms, is standard now and therefore we shall not go into the details.

We consider the following slightly more general metric

$$\begin{aligned} ds^2 &= 2(du + G\sin\alpha\, d\beta)(dr + H\sin\alpha\, d\beta) - M^2(d\alpha^2 + \sin^2\alpha\, d\beta^2) \\ &\quad - 2L(du + G\sin\alpha\, d\beta)^2 \qquad \ldots(2.1) \end{aligned}$$

with $M = M(\alpha, u, r)$.

Let us introduce the following basic 1-forms in the 4-dimensional Riemannian manifold defined by the metric (2.1).

$$\left.\begin{aligned} \theta^1 &= du + G\sin\alpha\, d\beta, & \theta^2 &= M d\alpha, \\ \theta^3 &= M\sin\alpha\, d\beta, & \theta^4 &= dr + H\sin\alpha\, d\beta - L\theta^1. \end{aligned}\right\} \qquad \ldots(2.2)$$

The metric (2.1) can now be expressed as

$$ds^2 = 2\theta^1\theta^4 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)}\theta^a\theta^b. \qquad ...(2.3)$$

With the help of the tetrad (2.2) and Cartan's equations of structure

$$d\theta^a = -\omega^a_{\ b} \wedge \theta^b \qquad ...(2.4)$$

$$d\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b} = \tfrac{1}{2} R^a_{\ bcd}\, \theta^c \wedge \theta^d \qquad ...(2.5)$$

we can obtain the tetrad components $R^a_{\ bcd}$ of the curvature tensor. From $R^a_{\ bcd}$ it is easy to obtain the tetrad components $R_{(ab)} = R^c_{\ abc}$ of Ricci tensor. The explicit expressions for $R_{(ab)}$ are

$$\begin{aligned}
R_{(23)} &= 0,\\
R_{(44)} &= (2/M)\,(M_{rr} - f^2/M^3),\\
R_{(24)} &= (G/M)\,[(M_r/M)_y - z_y(f/M^2)_r - (f/M^2)_u],\\
R_{(34)} &= -\,(G/M)\,[(M_r/M)_u + z_y(M_r/M)_r + (f/M^2)_y],\\
R_{(14)} &= (2/M)\,[M_{ru} + (LM_r)_r + (Lf^2/M^3) - (fh/M^3)] + L_{rr},\\
R_{(12)} &= LR_{(24)} + (G/M)\,[\{(2fL - H)/M^2\}_u\\
&\qquad + (L_r + M_u/M)_y + z_y\{(2fL - h)/M^2\}_r],\\
R_{(13)} &= LR_{(34)} + (G/M)\,[\{(2fL - h)/M^2\}_y\\
&\qquad - (L_r + M_u/M)_u - z_y(L_r + M_u/M)_r],\\
R_{(22)} &= R_{(33)} = (1/M^2)\,[G^2(M_u/M)_u + G^2(M_y/M)_y\\
&\qquad + 2f(M_y/M) - 1 + 4Lf^2/M^2 - 4fh/M^2 - (M^2)_{ur}\\
&\qquad - \{L(M^2)_r\}_r + H^2(M_r/M)_r + 2GH(M_r/M)_u],\\
R_{(11)} &= L^2R_{(44)} + (1/M^2)\,[G^2(L_{uu} + L_{yy}) + 2fL_y\\
&\qquad + 2ML_uM_r - 2ML_rM_u + 4LM\,M_{ru} + 2MM_{uu}\\
&\qquad + 2h^2/M^2 + 4fhL/M^2 + 2GHL_{ru} + H^2L_{rr}]
\end{aligned} \qquad ...(2.6)$$

with $2f = G_\alpha + G\cot\alpha$, $2h = H_\alpha + H\cot\alpha$ and a suffix denoting partial derivatives (e.g. $L_r = \partial L/\partial r$, $M_u = \partial M/\partial u$ etc.).

Here and in what follows the variables y and z are defined by the differential relations

$$G\,d\alpha = dy, \quad H\,d\alpha = dz.$$

3. The Field Equations

We shall now try to solve the field equations $R_{ik} = -8\pi\sigma\xi_i\xi_k$ where ξ_i is a null vector. The tetrad components of ξ_i are (1, 0, 0, 0) and so the above field equations will imply that $R_{(11)} = -8\pi\sigma$ and all other $R_{(ab)}$ will vanish.

The vanishing of $R_{(44)}$ gives us

$$M^2 = (f/Y)(X^2 + Y^2) \qquad \ldots(3.1)$$

where $Y = Y(u, y),\ X = X(r, u, y)$ with $X_r = -1$.

Making use of this value of M in the equations

$$R_{(24)} = 0 \text{ and } R_{(34)} = 0$$

we obtain

$$X_u = -(Y - z)_y,\ X_y = (Y - z)_u. \qquad \ldots(3.2)$$

In the next step we take the equation $R_{(22)} = 0$ and solve it to determine the form of the function $2L$. We shall find that

$$2L = -(Y_u/Y)X + 2J + \frac{2FX + 2EY}{X^2 + Y^2} \qquad \ldots(3.3)$$

with $J = J(u, y)$ given by

$$2J = (Y/f)[-\tfrac{1}{2}G^2\nabla^2 \log(Y/f) - 1 + f_y - 3fY_y/Y + 2fz_y/Y] \quad \ldots(3.4)$$

and

$$\left.\begin{aligned} &\nabla^2 = \partial^2/\partial u^2 + \partial^2/\partial y^2 \\ &E = -2YJ - YY_y + Yz_y + hY/f \end{aligned}\right\} \qquad \ldots(3.5)$$

F is an undetermined function of u and y. It is painless to verify that, with M given by (3.1) with (3.2) and $2L$ given by (3.3) with (3.4) and (3.5), the equation $R_{(14)} = 0$ is identically satisfied. Further the equations $R_{(12)} = 0$ and $R_{(13)} = 0$ will lead to

$$F_u = -E_y,\ F_y = E_u. \qquad \ldots(3.6)$$

A straightforward but lengthy calculation will now give us the following expression for $R_{(11)}$.

$$R_{(11)} = (1/M^2)[G^2\{\nabla^2 J - X_u(\log Y)_{uu} - X_y(\log Y)_{uy}\}$$
$$+ 2fJ_y - 2fF_u/Y + 3fFY_u/Y - 2fY_u^2/Y + fY_{uu} + GHY_u/Y]. \qquad \ldots(3.7)$$

Up to this stage we have worked with the general metric (2.1). In the next section we shall obtain radiating Demianski-type metrics and shall see that Demianski's metric comes out as a particular case.

4. The Radiating Demianski-type Metrics

For the metric (1.5) we have

$M^2 = X^2 + Y^2$ so that (3.1) gives

$$f = Y = Y(y). \qquad \ldots(4.1)$$

Equation (3.2) would then imply that

$$X = au - r, \;\; Y = -ay + z - b. \qquad \ldots(4.2)$$

Here a and b are constants of integration and no additional constant is shown explicitly in X because such a constant can always be incorporated in the r-coordinate. The result (3.4) then gives us

$$2J = 2a - 1. \qquad \ldots(4.3)$$

From the results (3.5) and (3.6) it follows that

$$E = E(y) = (1 - a)\, Y + h \qquad \ldots(4.4)$$

and

$$F = -\gamma u - m, \;\; E = \gamma y - (1 - a)\, b \qquad \ldots(4.5)$$

where γ and m are constants of integration.

From (4.4) and (4.5) we have

$$\gamma y = (1 - a)(-ay + z) + h. \qquad \ldots(4.6)$$

Thus the second equation of (4.2) and (4.6) are two equations for two unknown functions G and H.

The coefficients of the metric (2.1) are thus determined. The only non-vanishing components of $R_{(ab)}$ is $R_{(11)}$. The field equations $R_{ik} = -8\pi\sigma\xi_i\xi_k$, $\xi^i\xi_i = 0$ gives us the radiation density σ as

$$8\pi\sigma = -2\gamma/(X^2 + Y^2). \qquad \ldots(4.7)$$

With $G\, d\alpha = dy$, $H\, d\alpha = dz$, $2h = H_\alpha + H \cot\alpha$ and $2f = G_\alpha + G \cot\alpha$ the results (4.1), (4.2) and (4.6) show that the functions Y and H satisfy the following differential equations :

$$(1 - q^2)\, Y_{qq} - 2qY_q + 2aY = 2h \qquad \ldots(4.8)$$

$$(1 - q)\, h_{qq} - 2qh_q + 2(1 - a)\, h = 2\,[\gamma - a(a - 1)]\, Y \qquad \ldots(4.9)$$

where $q = \cos\alpha$.

If $h = 0$, then (4.9) gives us $\gamma = a(a - 1)$. In this case we obtain $H \sin\alpha =$ constant. If we choose this constant to be zero, then the metric (1.5) reduces to the metric considered by Vaidya, *et al.* (1976). We now consider the case in which $h \neq 0$.

If we set $p^2 = 1 + 4\gamma$ and $0 < p \leqslant 5/4$, then it can be easily seen that the differential eqns. (4.8) and (4.9) are equivalent to the equations

$$(1 - q^2) Z_{qq} - 2qZ_q + n(n+1) Z = 0 \quad \text{...(4.10)}$$

$$(1 - q^2) N_{qq} - 2qN_q + l(l+1) N = 0 \quad \text{...(4.11)}$$

with

$$1 - p = n(n+1), \quad 1 + p = l(l+1),$$
$$Z = h + \tfrac{1}{2}(1 - 2a + p) Y, \quad N = h + \tfrac{1}{2}(1 - 2a - p) Y. \quad \text{...(4.12)}$$

We need those solutions of (4.10) and (4.11) which will give us Demianski metric for $a = \gamma = 0$. The solutions of the eqns. (4.10) and (4.11) can be seen from any standard text like Coddington (1961). They are

$$Z = b(1 - a) Q_n(q), \quad N = kP_l(q) + cQ_l(q) \quad \text{...(4.13)}$$

where $P_l(q)$ and $Q_n(q)$ stand for the sums of the following infinite convergent series and k and c are arbitrary constants

$$P_l(q) = q - \frac{(l+2)(l-1)}{3!} q^3 + \frac{(l+4)(l+2)(l-1)(l-3)}{5!} q^5 - \dots \quad \text{...(4.14)}$$

$$Q_n(q) = 1 - \frac{(n+1)n}{2!} q^2 + \frac{(n+3)(n+1)n(n-2)}{4!} q^4 - \dots \quad \text{...(4.15)}$$

The series (4.14) and (4.15) are convergent for $q^2 < 1$. Knowing Z and N from (4.13) the result (4.12) will give Y and h as

$$pY = Z - N, \quad 2ph = (p - 1 + 2a) Z + (p + 1 - 2a) N. \quad \text{...(4.16)}$$

The functions G and H can now be determined from the relations

$$(G \sin \alpha)_q = -2Y \text{ and } (H \sin \alpha)_q = -2h.$$

They are given by

$$G \sin \alpha = -2 \int Y(q)\, dq, \quad H \sin \alpha = -2 \int h(q)\, dq. \quad \text{...(4.17)}$$

Here it should be noted that the functions $Q_n(q)$, $P_l(q)$, $Q_l(q)$ are continuous in the interval $(-1, 1)$ and therefore they are integrable in $(-1, 1)$.

One can therefore obtain the line-element in the final form as

$$ds^2 = 2\,[du - 2(\textstyle\int Y\, dq)\, d\beta]\,[dr - 2(\textstyle\int h\, dq)\, d\beta]$$
$$+ \left[1 - 2a - \frac{2FX + 2EY}{X^2 + Y^2}\right][du - 2(\textstyle\int Y\, dq)\, d\beta]^2$$
$$- (X^2 + Y^2)(d\alpha^2 + \sin^2 \alpha\, d\beta^2) \quad \text{...(4.18)}$$

with $X = au - r$, $E = h + (1 - a) Y$, $F = -\gamma u - m$ where h and Y are given by (4.13) and (4.16) alongwith (4.14) and (4.15). The metric (4.18) is the radiating Demianski-type metric.

The restriction $0 < p \leqslant 5/4$ is equivalent to $-1/4 < \gamma \leqslant 9/64$. If we choose γ to be negative i.e. $-1/4 < \gamma \leqslant 0$, it is clear from (4.7) that the radiation density σ is positive.

If $\gamma = 0$, then $p = 1$ and consequently $n = 0$ and $l = 1$. In this case the null radiation disappears and we get a vacuum space-time. For this vacuum space-time we have

$$X = au - r, \quad -Y = \bar{b} + k \cos\alpha + c \cos\alpha \log\tan\frac{\alpha}{2} + c,$$

$$G \sin\alpha = 2\bar{b} \cos\alpha - k \sin^2\alpha - c \sin^2\alpha \log\tan\frac{\alpha}{2} + c \cos\alpha$$

$$H \sin\alpha + (1 - a) G \sin\alpha = 2\bar{b} \cos\alpha \qquad \text{...(4.19)}$$

where $\bar{b} = b(1 - a)$. Further the function $2L$ for this vacuum space-time is given by

$$-2L = 1 - 2a - \frac{2m(r - au) - 2\bar{b}Y}{(r - au)^2 + Y^2} \qquad \text{...(4.20)}$$

where Y is given by (4.19). Now substituting $\bar{r} = r - au$ and using the results (4.19) and (4.20) the metric of the above-mentioned vacuum space-time becomes

$$\begin{aligned} ds = {} & 2(du + G \sin\alpha\, d\beta)\, [(d\bar{r} - (G \sin\alpha - 2\bar{b} \cos\alpha)\, d\beta] \\ & - (\bar{r}^2 + Y^2)\, (d\alpha^2 + \sin^2\alpha\, d\beta^2) \\ & + \left[1 - \frac{2m\bar{r} - 2\bar{b}Y}{\bar{r}^2 + Y^2}\right] (du + G \sin\alpha\, d\beta)^2 \end{aligned} \qquad \text{...(4.21)}$$

where Y and $G \sin\alpha$ are given by (4.19). It is not difficult to see that the metric (4.21) is the same as Demianski metric (1.3) with slight change of notations. Thus the metric (4.18) is the radiating version of the Demianski metric.

It should be noted that the Demianski metric and radiating Demianski-type metric have a singularity along the axis of symmetry (i.e. $\alpha = 0$ and $\alpha = \pi$).

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