LARGE DEFLECTIONS OF HEATED ORTHOTROPIC PLATES

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This paper presents an analysis of the large deflections of a heated orthotropic rectangular plate. The governing equations are derived on the basis of Berger's assumption and have been solved for a simply-supported plate.

1. Introduction

The classical large deflection plate problems usually lead to non-linear differential equations which cannot be exactly solved. Berger (1955) has shown that if, in deriving the differential equations from the expression for strain energy, the strain energy due to the second invariant in the middle surface of the plate is neglected, a simple fourth-order differential equation coupled with a non-linear second order equation is obtained. Basuli (1968) has extended the method to some isotropic elastic plates under stationary temperature distribution only. The present author has investigated several problems (Biswas 1974-76) relating to isotropic plates under stationary and non-stationary temperature distributions.

So far in the literature, problems on large deflections of orthotropic heated plates have not been considered at all. The present paper is devoted to determine the large deflection of a rectangular orthotropic plate under stationary temperature distribution.

2. Notations

u, v, w = displacement components

h = plate thickness

K_1, K_2, K_3 = thermal conductivities along the co-ordinate axes

$$\bar{e}_{xx} = \frac{\partial u}{\partial x} + \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\bar{e}_{yy} = \frac{\partial v}{\partial y} + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$\bar{e}_1 = \bar{e}_{xx} + K\bar{e}_{yy}$ = first strain invariant

Key words : Orthotropic, strain invariants, Berger's assumption, strain energy.

$$\left.\begin{aligned} e_{xx} &= \bar{e}_{xx} - z\frac{\partial^2 w}{\partial x^2} \\ e_{yy} &= \bar{e}_{yy} - z\frac{\partial^2 w}{\partial y^2} \end{aligned}\right\} = \text{thermal strain components}$$

α_1, α_2 = coefficients of thermal expansions

$$\beta_1 = \frac{\alpha_2 S_{12} - \alpha_1 S_{22}}{S_{11}S_{22} - S_{12}^2}$$

$$\beta_2 = \frac{\alpha_1 S_{12} - \alpha_2 S_{11}}{S_{11}S_{22} - S_{12}^2}$$

α^2 = normalized constant of integration

$$M_T = \int_{-h/2}^{h/2} zT\,dz$$

$T = T(x, y, z)$ = temperature distribution at any point (x, y, z)

$S_{ij}, E'_x, E'_y, E'', G$ = elastic constants

$$D_x = \frac{E'_x h^3}{12},\ D_y = \frac{E'_y h^3}{12},\ D_{xy} = \frac{Gh^3}{12}$$

$$D_1 = \frac{E''h^3}{12},\ H = D_1 + 2D_{xy},\ K = \sqrt{\frac{D_y}{D_x}}.$$

3. Governing Equations

Combining the potential energy due to bending and stretching of the middle surface of an orthotropic plate undergoing large deflection ($\bar{e}_2$ being neglected, which is Berger's assumption) with the potential energy due to heating one gets the total potential energy, V, as

$$V = \tfrac{1}{2}\iint_S \left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{xy}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + D_x\frac{12}{h^2}\bar{e}_1^2\right] dx\,dy + \iint_S\int_{-h/2}^{h/2} [\beta_1 e_{xx}T(x, y, z) + \beta_2 e_{yy}T(x, y, z)]\,dx\,dy\,dz. \quad ...(1)$$

In the steady state the temperature $T(x, y, z)$ at any point of an orthotropic material has to satisfy the Fourier heat equation

$$K_1 \frac{\partial^2 T}{\partial x^2} + K_2 \frac{\partial^2 T}{\partial y^2} + K_3 \frac{\partial^2 T}{\partial z^2} = 0. \qquad ...(2)$$

A solution of eqn. (2) can be taken as

$$T(x, y, z) = 2 \left(\frac{z}{h} \right) \theta, \; \theta = \text{constant} \qquad ...(3)$$

compatible with the boundary conditions

$$T\left(x, y, \frac{h}{2}\right) = \theta \text{ and } T\left(x, y, \frac{-h}{2}\right) = -\theta. \qquad ...(4)$$

Combining (1) and (3) and using the following Euler's equations of the calculus of variations

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial u_y} = 0 \qquad ...(5)$$

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x} \frac{\partial F}{\partial v_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial v_y} = 0 \qquad ...(6)$$

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x} \frac{\partial F}{\partial w_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial w_y} + \frac{\partial^2}{\partial x^2} \frac{\partial F}{\partial w_{xx}} + \frac{\partial^2}{\partial x \partial y} \frac{\partial F}{\partial w_{xy}} + \frac{\partial^2}{\partial y^2} \frac{\partial F}{\partial w_{yy}} = 0 \qquad ...(7)$$

one gets

$$\frac{\partial}{\partial x} (\bar{e}_1) = 0 \qquad ...(8)$$

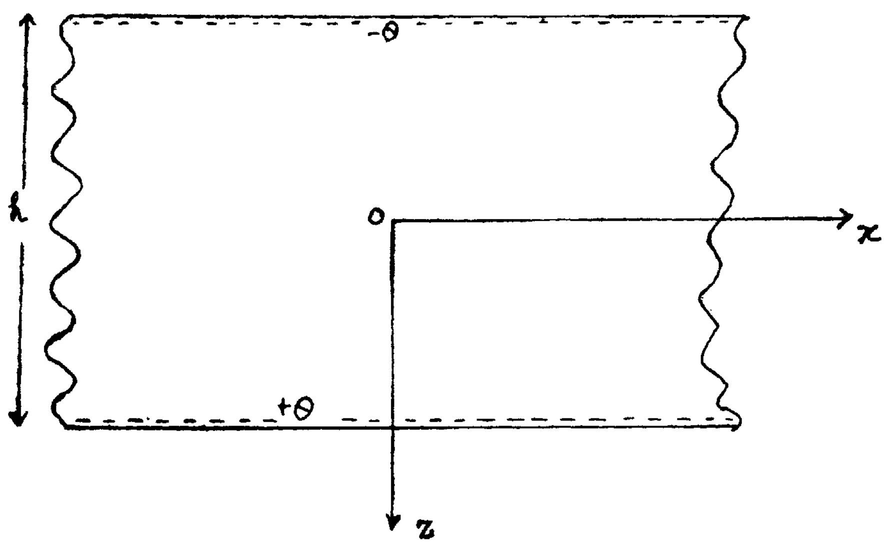

FIG. 1. Plate Geometry

$$\frac{\partial}{\partial y}(\bar{e}_1) = 0 \qquad \ldots(9)$$

$$\frac{\partial^2}{\partial x^2}(D_x w_{xx} + D_1 w_{yy} + \beta_1 M_T) + \frac{\partial^2}{\partial y^2}(D_y w_{yy} + D_1 w_{xx} + \beta_2 M_T)$$
$$+ \frac{\partial^2}{\partial x^2}(4D_{xy} w_{xy})$$
$$- \frac{\partial}{\partial x}\left\{D_x \frac{12}{h^2} e_1 w_x\right\} - \frac{\partial}{\partial y}\left\{D_x \frac{12}{h^2} e_1 K w_y\right\} = 0. \qquad \ldots(10)$$

Now from (8) and (9) one gets

$$\bar{e}_1 = \text{constant} = \frac{\alpha^2 h^2}{12} \text{ (say)}. \qquad \ldots(11)$$

Also from (10) and (11) one gets

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - D_x \alpha^2 \left(\frac{\partial^2 w}{\partial x^2} + K \frac{\partial^2 w}{\partial y^2}\right)$$
$$+ \beta_1 \frac{\partial^2 M_T}{\partial x^2} + \beta_2 \frac{\partial^2 M_T}{\partial y^2} = 0. \qquad \ldots(12)$$

4. Method of Solution for a Simply-supported Rectangular Plate

For a rectangular plate of sides a and b satisfying simply-supported boundary conditions, the deflection w can be expressed in the form

$$w = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad \ldots(13)$$

Since $$M_T = \frac{\theta}{6} h^2 = \text{constant} = M_0 \text{ (say)} \qquad \ldots(14)$$

one can express it in the form of the double Fourier series

$$M_T = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{16 M_0}{mn\pi^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{6}. \qquad \ldots(15)$$

Substituting (13) and (15) into (12) one gets

$$A_{mn} = \frac{\frac{16 M_0}{mn\pi^2}\left(\beta_1 \frac{m^2\pi^2}{a^2} + \beta_2 \frac{n^2\pi^2}{b^2}\right)}{D_x \frac{m^4\pi^4}{a^4} + 2H \frac{m^2 n^2 \pi^2}{a^2 b^2} + D_y \frac{n^4\pi^4}{b^4} + D_x \alpha^2 \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\sqrt{\frac{D_y}{D_x}}\right)}. \qquad \ldots(16)$$

We are interested only in the normal displacement, and therefore to determine w and to eliminate u and v suitable forms of them compatible with the boundary conditions

$$\left.\begin{aligned} u &= 0 \text{ on } x = 0, a \\ v &= 0 \text{ on } y = 0, b \end{aligned}\right\} \qquad \text{...(17)}$$

are assumed as

$$u = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \chi_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \qquad \text{...(18)}$$

$$v = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \chi_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \qquad \text{...(19)}$$

Substituting (18), (19) and (13) into (11) and integrating over the area of the plate one gets

$$\sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \tfrac{1}{8} A_{mn}^2 \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right) = \frac{\alpha^2 h^2}{12}. \qquad \text{...(20)}$$

Combining (16) and (20), eqn. (13) determines the deflection completely.

As $\alpha \to 0$ into eqn. (13) one gets the corresponding small deflection w for orthotropic plate as given by Misra (1973). Also for an isotropic plate and when $\alpha \to 0$ one gets the corresponding small deflection as given in Nowacki [1962, p. 402, Art 15.5(*c*)].

5. Numerical Results

In finding the deflection at a point one has to start from eqn. (20) with an assumed value of αa leading to a particular value of the temperature parameter $\theta \left(\frac{a}{h}\right)^2 \frac{\beta_1 h^3}{D_x}$. For this value of the temperature parameter eqn. (13) finally gives the corresponding deflection. The following data are used for computing central deflections and have been plotted graphically showing the variation of deflection with temperature (Fig. 2).

$$\frac{D_y}{D_x} = 0.32, \; \frac{H}{D_x} = 0.2, \; \frac{a}{h} = 20, \; \frac{\beta_2}{\beta_1} = 0.5.$$

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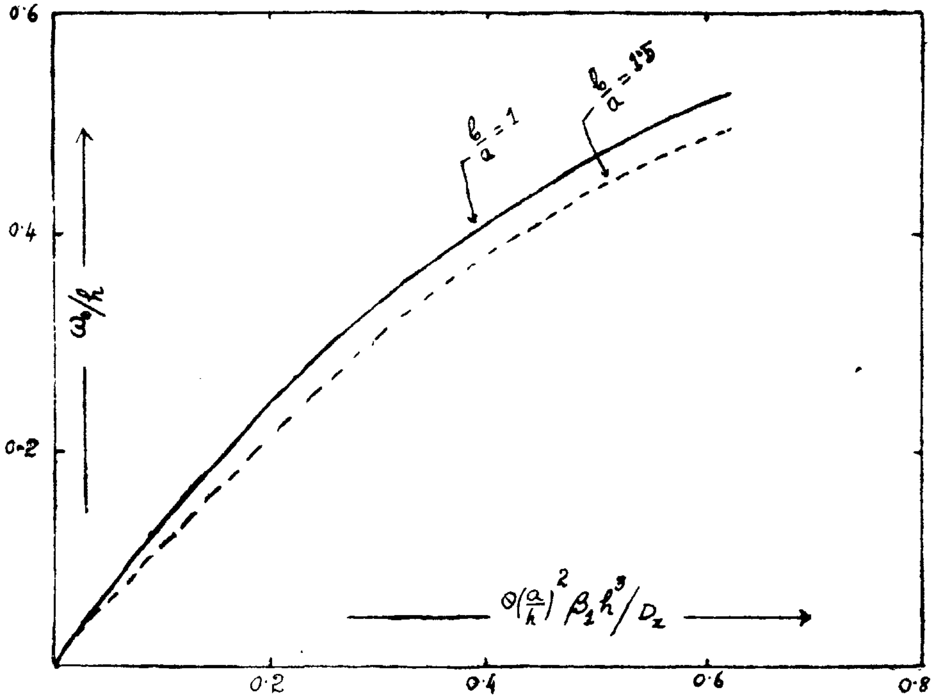

FIG. 2. Variation of central deflections with temperature parameter.

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