ON AN INVARIANT SUBMANIFOLD IN A $f(5, 3)$-STRUCTURE MANIFOLD

by R. N. Singh *and* R. S. Mishra, *Department of Mathematics, Banaras Hindu University, Varanasi* 220005

(*Received* 3 *January* 1978)

In this paper, we have defined an invariant submanifold in a $f(5, 3)$-structure manifold and studied its properties in relation with the manifold.

1. Introduction

Let M_n be an n-dimensional C^∞ manifold equipped with a non-zero (1, 1) type tensor field $\tilde{f}$ satisfying

$$\tilde{f}^5 + \tilde{f}^3 = 0 \qquad \text{...(1.1a)}$$

and

$$\text{rank } (\tilde{f}) = \tfrac{1}{2} (\text{rank } \tilde{f}^2 + \text{dimension } M_n)$$

$$= \tfrac{1}{3} (\text{rank } \tilde{f}^3 + 2 \text{ times dimension } M_n). \qquad \text{...(1.1b)}$$

Then the structure is called $f(5, 3)$-structure and the manifold M_n, equipped with this structure, is named as $f(5, 3)$-structure manifold (Ishihara and Yano 1964).

Let V_m be a C^∞ m-dimensional manifold embedded as a submanifold in M_n (with a tensor field $\tilde{f}$ of type (1,1)). Let ϕ be the embedding defined by

$$\phi : V_m \rightarrow M_n. \qquad \text{...(1.2)}$$

Let B be the mapping induced by ϕ, i.e., $B = d\phi$,

$$B : T(V) \rightarrow T(M) \qquad \text{...(1.3)}$$

where $T(V)$ and $T(M)$ are tangent spaces of V_m and M_n at $p \in V_m$ and $q \in M_n$. Let $T(V, M)$ be the set of all tangent vectors to the submanifold $\phi(V)$. The mapping B given by

$$B : T(V) \rightarrow T(V, M)$$

is an isomorphism. The set of all vectors normal to $\phi(V)$ form a vector bundle over $\phi(V)$, which is denoted by $N(V, M)$ and the vector bundle induced by ϕ from $N(V, M)$ is denoted by $N(V)$. We denote the natural isomorphism by

$$C : N(V) \rightarrow N(V, M)$$

and the space of all C^∞ tensor fields of type (r, s) associated with $N(V)$ by $\eta^r_s\,(V)$. Thus $\eta^0_0\,(V) = Z^0_0\,(V)$ is the space of all C^∞ functions defined on V_m, while an element of $\eta^1_0\,(V)$ is a C^∞ vector field, normal to V_m and an element of $Z^1_0\,(V)$ is a C^∞ vector field tangential to V_m.

Let us consider vector fields $\hat{X}$ and $\hat{Y}$ along $\phi(V)$. Let $\tilde{X}$ and $\tilde{Y}$ be the local extensions on $\hat{X}$ and $\hat{Y}$ respectively. Then $[\tilde{X}, \tilde{Y}]$ is a vector field tangential to M_n and its restriction $[\tilde{X}, \tilde{Y}]/\phi(V)$ to $\phi(V)$ is determined independently of the choice of these local extensions $\tilde{X}$ and $\tilde{Y}$. Therefore, we define $[\hat{X}, \hat{Y}]$ by

$$[\hat{X}, \hat{Y}] \overset{def}{=} [\tilde{X}, \tilde{Y}]/\phi(V).$$

Since B is an isomorphism, we have $[BX, BY] = B\,[X, Y]$ for all $X \in Z^1_0\,(V)$ and $Y \in Z^1_0\,(V)$, where $Z^r_s\,(V)$ is the space of all C^∞ tensor fields of type (r, s) associated with $T(V)$.

Definition — An m-dimensional C^∞ manifold V_m is defined to be an invariant submanifold of M_n, if the tangent space $T_{p'}(\phi(V))$ of $\phi(V)$ is invariant by the linear mapping f at each point p' of $\phi(V)$.

For vector fields $X, Y, Z, \ldots$ etc. in V_m, we have

$$\tilde{f}(BX) = B(f_1(X)) \qquad \ldots(1.4)$$

where $\tilde{f}_1$ is a $f(5, 3)$-structure and f_1 is a tensor of type $(1, 1)$ in V_m.

2. Nijenhuis Tensors

Let us denote by $\tilde{N}_1$ and N_1, the Nijenhuis tensors in M_n and V_m, determined by $\tilde{f}_1$ and f_1 respectively.

Theorem 2.1 — The Nijenhuis tensors $\tilde{N}_1$ and N_1 are related by

$$\tilde{N}_1(BX, BY) = B(N_1(X, Y)).$$

Proof : $\tilde{N}_1(\tilde{X}, \tilde{Y}) = [\tilde{f}_1\tilde{X}, \tilde{f}_1\tilde{Y}] - \tilde{f}_1\,[\tilde{f}_1\tilde{X}, \tilde{Y}] - \tilde{f}_1\,[\tilde{X}, \tilde{f}_1\tilde{Y}] + \tilde{f}_1^2\,[\tilde{X}, \tilde{Y}]$

The complementary projection operators are

$$\tilde{l}_1 = \tilde{f}_1^4, \;\; \tilde{m}_1 = I - \tilde{f}_1^4. \qquad \ldots(2.1)$$

Let $D_{\tilde{l}_1}$ and $D_{\tilde{m}_1}$ be the distributions corresponding to the complementary projection operators $\tilde{l}_1$ and $\tilde{m}_1$ respectively. If the rank of f is r, then $D_{\tilde{l}_1}$ is r-dimensional and $D_{\tilde{m}_1}$ is $(n - r)$-dimensional.

Let us consider the following two cases for an invariant submanifold V_m in a generalized f-structure manifold M_n.

Case 1 — Let us assume that the distribution $D_{\tilde{m}_1}$ is never tangential to $\phi(V)$, that is, no vector field of the type $\tilde{m}_1(\tilde{X})$ (where $\tilde{X}$ is a vector field tangential to $\phi(V)$) is tangential to $\phi(V)$. Later, it will be proved that in the case V_m is necessarily even dimensional.

Applying $\tilde{f}_1$ to (1.4), we get

$$\tilde{f}_1^2(B(X)) = B(f_1^2(X)). \qquad \ldots(2.2)$$

We now show that the vector fields of type BX are in the distribution $D_{\tilde{l}_1}$, which is equivalent to showing that $\tilde{m}_1(BX) = 0$.

Suppose that $\tilde{m}_1(BX) \neq 0$.

In view of (2.1), we have

$$\begin{aligned}\tilde{m}_1(BX) &= (I - \tilde{f}_1^4)\, BX \\ &= BX - \tilde{f}_1^4(BX) \\ &= BX - B(f_1^4(X)) \\ &= B(X - f_1^4(X)) \\ &= B(I - f_1^4)\, X.\end{aligned}$$

The above relation shows that $\tilde{m}_1(BX)$ is tangential to $\phi(V)$, which contradicts the hypothesis. Hence

$$\tilde{m}_1(BX) = 0.$$

Since, $\tilde{f}_1$ acts on $D_{\tilde{l}_1}$ as an almost complex structure, hence

$$Bf_1^2(X) = -\, BX$$

which in view of B being isomorphism, yields

$$f_1^2(X) = -\, X. \qquad \ldots(2.3)$$

Consequently, the (1, 1) tensor field f_1 in V_m is an almost complex structure, called induced almost complex structure on the invariant submanifold V_m.

3. Haantje's Tensor

We define Haantje's tensor (Yano 1963) as a tensor field $\tilde{H}_1$ of type (1, 2) in M_n as follows :

$$\tilde{H}_1(\tilde{X}, \tilde{Y}) \overset{def}{=} \tilde{N}_1(\tilde{X}, \tilde{Y}) - \tilde{N}_1(\tilde{m}\tilde{X}, \tilde{Y}) - \tilde{N}_1(\tilde{X}, \tilde{m}\tilde{Y}) + \tilde{N}_1(\tilde{m}\tilde{X}, \tilde{m}\tilde{Y}). \quad ...(3.1)$$

Theorem 3.1 — The tensor field $\tilde{H}_1$ of type (1, 2) defined in M_n satisfies

$$\tilde{H}_1(BX, BY) = \tilde{N}_1(BX, BY) = B(N_1(X, Y)). \quad ...(3.2)$$

Proof : Since any vector field tangential to $\phi(V)$ is not contained in the distribution $D_{\tilde{m}_1}$, we have

$$\tilde{m}_1(BX) = 0 \quad ...(3.3)$$

where X is any vector field in V_m. In consequence of Theorem 2.1 and eqns. (3.1) and (3.3), we see that

$$\tilde{H}_1(BX, BY) = B(N_1(X, Y)).$$

Combining the above results, we can state the following :

Theorem 3.2 — An invariant submanifold V_m embedded in a $f(5, 3)$-structure manifold M_n such that the distribution $D_{\tilde{m}}(D_{\tilde{m}_1})$ is never tangential to $\phi(V)$, is an almost complex manifold with induced almost complex structure f. Consequently the dimension of V_m is even. In generalized f-structure manifold M_n Haantje's tensor vanishes if and only if the invariant submanifold V_m is complex.

Case 2 — The distribution $D_{\tilde{m}_1}$ is always tangential to the invariant submanifold $\phi(V)$ implies for each vector field X in V_m

$$\tilde{m}_1(BX) = B(m_1(X)) \quad ...(3.4)$$

where m_1 is a (1, 1) tensor field in V_m given by

$$m_1 = I - f^4.$$

Again, we define a (1, 1) tensor field l_1 in V_m by

$$l_1 \overset{def}{=} f^4.$$

Thus

$$l_1(X) = f_1^4(X)$$

for all vector fields X in the tangent space of V_m. Applying B on both sides of the above expression, we get

$$\begin{aligned} B(l_1(X)) &= B(f_1^4(X)) \\ &= \tilde{f}_1^4(B(X)) \\ &= \tilde{l}_1(B(X)). \end{aligned} \qquad \therefore(3.5)$$

Theorem 3.3 — The tensor fields l_1 and m_1 of type (1, 1) in V_m defined by (3.4) and (3.5) satisfy

$$\begin{aligned} & l_1 + m_1 = I, \\ & l_1 m_1 = m_1 l_1 = 0, \\ & l_1^2 = l_1,\ m_1^2 = m_1. \end{aligned} \qquad ...(3.6)$$

PROOF : Since $\tilde{l}_1 + \tilde{m}_1 = I$, then operating it on a vector of type BX, we get

$$\tilde{l}_1(BX) + \tilde{m}_1(BX) = BX.$$

Hence,

$$B(l_1(X)) + B(m_1(X)) = BX.$$

Since B is an isomorphism, hence

$$l_1 + m_1 = I$$

which proves first relation of (3.6). Similarly, we can prove the others. Moreover,

$$\begin{aligned} B(f_1^5(X)) &= f_1^5(BX) \\ &= -\tilde{f}_1^3(BX) = -B(f_1^3(X)). \end{aligned}$$

Hence,

$$f_1^5 = f_1^3 = 0.$$

Theorem 3.4 — We have

$$\tilde{H}_1(BX, BY) = BH_1(X, Y).$$

PROOF : Writing BX for X and BY for Y in (3.1), we get the result.

Thus we have the following.

Theorem 3.5 — An invariant submanifold V_m embedded in a $f(5, 3)$-structure manifold M_n in such a way that the distribution $D\tilde{m}_1$ is always tangential to $\phi(V)$ is a $f(5, 3)$-structure manifold with the induced structure f_1. The Haantje's tensor vanishes in M_n if and only if it vanishes in V_m.

REFERENCES

Ishihara, S., and Yano, K. (1964). On integrability conditions of a structure f satisfying $f^3 + f = 0$. *Quart. J. Math.*, **15**, 217-22.

Yano, K. (1963). On a structure defined by a tensor field f of type (1, 1) satisfying $f^3 + f = 0$. *Tensor, N. S.*, **14**, 99-109.