LIE DERIVATIVE IN SPECIAL KAWAGUCHI SPACE

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We consider an n-dimensional special Kawaguchi space $K_n^{(1)}$ and obtain the lie derivatives of $X^i(x, x', \xi)$ and $T^{ij}(x, x', \xi)$ by taking the deformation vector as a function of positional and directional arguments both. The lie derivative of the connection parameter is also obtained.

1. INTRODUCTION

Let us consider an n-dimensional space V_n endowed with a coordinate system x^i $(i = 1, 2, \ldots n)$ and let a curve $x^i = x^i(t)$ be defined in this space. If the arc length of the curve is given by the integral

$$s = \int \{A_i(x, x')\, x''^i + B(x, x')\}^{1/p}\, dt, \qquad \ldots(1.1)$$

where $x'^i = dx^i/dt$, $x''^i = d^2x^i/dt^2$, and A_i and B being differentiable functions of x^i and x'^i then the space V_n is called special Kawaguchi space $K_n^{(1)}$ (Kawaguchi 1936, 1938).

The space with the arc length (1.1) satisfies the following relations :

$$\left.\begin{array}{ll}
\text{(a)}\quad A_{k(i)}\, x'^i = (p-2)\, A_k, & \text{(b)}\quad B_{(i)} x'^i = pB, \\
\text{(c)}\quad G_{ik} = 2A_{i(k)} - A_{k(i)}, & \text{(d)}\quad G_{ik} G^{il} = \delta^l_k, \\
\text{(e)}\quad \overset{[2]l}{x} = -\, T_i G^{il} = x''^l + 2\Gamma^l, & \text{(f)}\quad 2\Gamma^l = (2A_{ik} x'^k - B_{(i)})\, G^{il}, \\
\text{(g)}\quad T_i = (A_{k(i)} - 2A_{i(k)})\, x''^k - 2A_{il} x'^l + B_{(i)}. &
\end{array}\right\} \qquad \ldots(1.2)$$

The covariant derivative (Kawaguchi 1938) of a vector $X^i(x, x')$ in $K_n^{(1)}$ is given by $\nabla_j X^i = \partial X^i/\partial x^j - \partial X^i/\partial x'^k\, \Gamma^k_{(j)} + \Gamma^i_{(k)(j)} X^k$ in the direction of x'^k, where

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$X^i(x, x')$ is homogeneous function of degree zero with regards to x'^k. If the vector field $X^i(x, x', \xi)$ is dependent on positional coordinate and directional arguments then the covariant derivative at x^k in the direction of x'^k may be defined as (assuming the direction $\xi^k(x)$ stationary at the point considered i.e. $\nabla_j \xi^i = 0$)

$$\nabla_j X^i(x, x', \xi) = \frac{\partial X^i}{\partial x^j} + \frac{\partial X^i}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^j} - \frac{\partial X^i}{\partial x'^k}\Gamma^k_{(j)} + \Gamma^i_{(k)(j)} X^k. \quad \text{...(1.3)}$$

2. Infinitesimal Transformation and Lie Derivative

We consider an infinitesimal transformation

$$\bar{x}^i = x^i + v^i(x, \xi)\, d\tau, \quad \text{...(2.1)}$$

where the deformed vector $v^i(x, \xi)$ is a function of positional coordinate x^k and direction $\xi^k(x)$ defined over the region of the space considered and $d\tau$ is an infinitesimal constant. Under the transformation (2.1) the point x^i makes a small displacement $dx^i = v^i(x, \xi)\, d\tau$.

Differentiating (2.1) partially with respect to x^j and $\bar{x}^j$ respectively and by using $\frac{\partial v^i}{\partial \bar{x}^j} = \frac{\partial v^i}{\partial x^j}$ (Rund 1959), we obtain

$$\frac{\partial \bar{x}^i}{\partial x^j} = \delta^i_j + \left(\frac{\partial v^i}{\partial x^j}\right) d\tau \quad \text{...(2.2)}$$

and

$$\frac{\partial x^i}{\partial \bar{x}^j} = \delta^i_j - \left(\frac{\partial v^i}{\partial \bar{x}^j}\right) d\tau. \quad \text{...(2.3)}$$

Again, differentiating (2.1) with respect to t (parameter of the curve) we get

$$\bar{x}'^i = x'^i + \left\{\left(\frac{\partial v^i}{\partial x^j}\right) x'^j + \left(\frac{\partial v^i}{\partial \xi^j}\right) \xi'^j\right\} d\tau \quad \text{...(2.4)}$$

and

$$\bar{x}''^i = x''^i + \left\{\left(\frac{\partial^2 v^i}{\partial x^j \partial x^k}\right) x'^j x'^k + \left(\frac{\partial^2 v^i}{\partial \xi^j \partial \xi^k}\right) \xi'^j \xi'^k + 2\left(\frac{\partial^2 v^i}{\partial \xi^k \partial x^j}\right) \xi'^k x'^j \right.$$
$$\left. + \left(\frac{\partial v^i}{\partial x^j}\right) x''^j + \left(\frac{\partial v^i}{\partial \xi^j}\right) \xi''^j\right\} d\tau. \quad \text{...(2.5)}$$

The corresponding variation in the direction $\xi^i(x^k)$ is given by

$$\bar{\xi}^i = \xi^i + \left\{\left(\frac{\partial v^i}{\partial x^j}\right) \xi^j + \left(\frac{\partial v^i}{\partial \xi^j}\right) d\xi^j\right\} d\tau. \quad \text{...(2.6)}$$

The value of the vector field $X^i(x, x', \xi)$ at $(\bar{x}, \bar{x}'\ \bar{\xi})$ (taking first order in $d\tau$) is given by

$$X^i(\bar{x}, \bar{x}', \bar{\xi}) = X^i(x, x', \xi) + \left[\left(\frac{\partial X^i}{\partial x^k}\right) v^k + \left(\frac{\partial X^i}{\partial x'^k}\right)\left\{\left(\frac{\partial v^k}{\partial x^h}\right) x'^h + \left(\frac{\partial v^k}{\partial \xi^h}\right)\xi'^h\right\} + \left(\frac{\partial X^i}{\partial \xi^k}\right)\left\{\left(\frac{\partial v^k}{\partial x^h}\right) \xi^h + \left(\frac{\partial v^k}{\partial \xi^h}\right) d\xi^h\right\}\right] d\tau. \quad ...(2.7)$$

If we denote by $\bar{X}^i(\bar{x}, \bar{x}', \bar{\xi})$ the transform vector field of $X^i(x, x', \xi)$ from (x, x', ξ) to $(\bar{x}, \bar{x}', \bar{\xi})$ under the infinitesimal transformation (2.1) then we have

$$\bar{X}^i(\bar{x}, \bar{x}', \bar{\xi}) = X^i(x, x', \xi) + \left(\frac{\partial v^i}{\partial x^k}\right) X^k(x, x', \xi)\, d\tau. \quad ...(2.8)$$

Again, if we denote by $X^i(x, x', \xi \,\|\, \bar{x}, \bar{x}', \bar{\xi})$ the vector transported by parallism from (x, x', ξ) to $(\bar{x}, \bar{x}', \bar{\xi})$ then we have

$$\delta X^i = dX^i + \Gamma^i_{(j)(k)}(x, x', \xi)\, X^j dx^k = 0 \quad ...(2.9)$$

where δX^i is the absolute differential of the vector X^i.

Using Schouten's notations (Schouten and Van Kampen 1933) in eqns. (2.7), (2.8) and (2.9), we obtain

$$\overset{1}{d}X^i(x, x', \xi) = \left[\left(\frac{\partial X^i}{\partial x^k}\right) v^k + \left(\frac{\partial X^i}{\partial x'^k}\right)\left\{\left(\frac{\partial v^k}{\partial x^h}\right) x'^h + \left(\frac{\partial v^k}{\partial \xi^h}\right) \xi'^h\right\} + \left(\frac{\partial X^i}{\partial \xi^k}\right)\left\{\left(\frac{\partial v^k}{\partial x^h}\right) \xi^h + \left(\frac{\partial v^k}{\partial \xi^l}\right) d\xi^l\right\}\right] d\tau \quad ...(2.10)$$

$$\overset{2}{d}X^i(x, x', \xi) = \left(\frac{\partial v^i}{\partial x^k}\right) X^k(x, x', \xi)\, d\tau \quad ...(2.11)$$

and

$$\overset{3}{d}X^i(x, x', \xi) = -\,\Gamma^i_{(j)(k)}(x, x', \xi)\, X^j v^k\, d\tau. \quad ...(2.12)$$

We have the invariant operators (Davies 1939) given by

$$\overset{(r,s)}{D} \doteq \frac{\overset{r}{d} - \overset{s}{d}}{d\tau}, \; r, s = 1, 2, 3; \quad r \neq s. \quad ...(2.13)$$

Accordingly, we have the following :

Theorem 2.1 — For the vector field $X^i(x, x', \xi)$ of the space $K^{(1)}_n$ under the infinitesimal transformation (2.1), we have

$$\overset{(1,2)}{D} X^i(x, x', \xi) = (\nabla_k X^i)\, v^k + \left(\frac{\partial X^i}{\partial \xi^k}\right)\left\{(\nabla_j v^k)\, \xi^j + \left(\frac{\partial v^k}{\partial \xi^m}\right) d\xi^m - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right) \xi^m + \left(\frac{\partial \xi^k}{\partial x^m}\right) v^m\right\} +$$

(*equation continued on p.* 1042)

$$+ \left(\frac{\partial X^i}{\partial x'^k}\right)\left\{(\nabla_j v^k)\, x'^j - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^j}\right) x'^j + \left(\frac{\partial v^k}{\partial \xi^h}\right) \xi'^h\right\} - X^k \left\{(\nabla_k v^i) - \left(\frac{\partial v^i}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^k}\right)\right\}, \quad ...(2.14)$$

$$\overset{(1,3)}{D} X^i(x, x', \xi) = (\nabla_k X^i)\, v^k + \left(\frac{\partial X^i}{\partial \xi^k}\right)\left\{(\nabla_j v^k)\, \xi^j + \left(\frac{\partial \xi^k}{\partial x^j}\right) v^j + \left(\frac{\partial v^k}{\partial \xi^m}\right) d\xi^m - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right)\xi^m\right\} + \left(\frac{\partial X^i}{\partial x'^k}\right) \times \left\{(\nabla_j v^k)\, x'^j - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^j}\right) x'^j + \left(\frac{\partial v^k}{\partial \xi^h}\right)\xi'^h\right\} \quad ...(2.15)$$

and

$$\overset{(2,3)}{D} X^i(x, x', \xi) = \left\{(\nabla_k v^i) - \left(\frac{\partial v^i}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^k}\right)\right\} X^k. \quad ...(2.16)$$

PROOF : Equations (2.10), (2.11) and (2.12) yield three derivatives (2.14), (2.15) and (2.16) using the definition (2.13) and the covariant derivative (1.3).

Theorem 2.2 — If $T^{ij}(x, x', \xi)$ is the contravariant component of a second order tensor. Then we have

$$\overset{(1,2)}{D} T^{ij}(x, x', \xi) = (\nabla_k T^{ij})\, v^k + \left(\frac{\partial T^{ij}}{\partial \xi^k}\right)\left\{(\nabla_h v^k)\, \xi^h + \left(\frac{\partial v^k}{\partial \xi^h}\right) d\xi^h - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right)\xi^m + \left(\frac{\partial \xi^k}{\partial x^h}\right) v^h\right\} + \left(\frac{\partial T^{ij}}{\partial x'^k}\right) \times \left\{(\nabla_h v^k)\, x'^h + \left(\frac{\partial v^k}{\partial \xi^h}\right) \xi'^h - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right) x'^m\right\} - T^{hj}\left\{(\nabla_h v^i) - \left(\frac{\partial \xi^k}{\partial x^h}\right)\left(\frac{\partial v^i}{\partial \xi^k}\right)\right\} - T^{ih}\left\{(\nabla_h v^j) - \left(\frac{\partial \xi^k}{\partial x^h}\right)\left(\frac{\partial v^j}{\partial \xi^k}\right)\right\}, \quad ...(2.17)$$

$$\overset{(1,3)}{D} T^{ij}(x, x', \xi) = (\nabla_k T^{ij})\, v^k + \left(\frac{\partial T^{ij}}{\partial \xi^k}\right)\left\{(\nabla_h v^k)\, \xi^h + \left(\frac{\partial v^k}{\partial \xi^h}\right) d\xi^h - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right)\xi^m + \left(\frac{\partial \xi^k}{\partial x^m}\right) v^m\right\} + \left(\frac{\partial T^{ij}}{\partial x'^k}\right) \times \left\{(\nabla_h v^k)\, x'^h - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right) x'^m + \left(\frac{\partial v^k}{\partial \xi^m}\right) \xi'^m\right\} \quad ...(2.18)$$

and

$$\overset{(2,3)}{D}T^{ij}(x, x', \xi) = T^{ih}\left\{(\nabla_h v^j) - \left(\frac{\partial v^j}{\partial \xi^k}\right)\left(\frac{\partial \xi^k}{\partial x^h}\right)\right\}$$

$$+ T^{hj}\left\{(\nabla_h v^i) - \left(\frac{\partial v^i}{\partial \xi^k}\right)\left(\frac{\partial \xi^k}{\partial x^h}\right)\right\}. \quad \ldots(2.19)$$

Proof : If we denote by $T^{ij}(\bar{x}, \bar{x}', \bar{\xi})$ the deformed component of $T^{ij}(x, x', \xi)$ at $(\bar{x}, \bar{x}', \bar{\xi})$ under the infinitesimal transformation (2.1) then we get

$$T^{ij}(\bar{x}, \bar{x}', \bar{\xi}) = T^{ij}(x, x', \xi) + \left[\left(\frac{\partial T^{ij}}{\partial x^k}\right) v^k + \left(\frac{\partial T^{ij}}{\partial x'^k}\right)\left\{\left(\frac{\partial v^k}{\partial x^m}\right) x'^m\right.\right.$$

$$\left.\left. + \left(\frac{\partial v^k}{\partial \xi^m}\right) \xi'^m\right\} + \left(\frac{\partial T^{ij}}{\partial \xi^k}\right)\left\{\left(\frac{\partial v^k}{\partial x^m}\right) \xi^m + \left(\frac{\partial v^k}{\partial \xi^m}\right) d\xi^m\right\}\right] d\tau. \quad \ldots(2.20)$$

Again, if $\bar{T}^{ij}(\bar{x}, \bar{x}', \bar{\xi})$ denotes the transform of the tensor field $T^{ij}(x, x', \xi)$ in $\bar{x}^i$ coordinate system by considering the eqn. (2.1) as the coordinate transformation and using (2.2) then we obtain

$$\bar{T}^{ij}(\bar{x}, \bar{x}', \bar{\xi}) = T^{ij}(x, x', \xi) + \left\{T^{ik}\left(\frac{\partial v^j}{\partial x^k}\right) + T^{kj}\left(\frac{\partial v^i}{\partial x^k}\right)\right\} d\tau. \quad \ldots(2.21)$$

On the other hand, if we suppose $T^{ij}(x, x', \xi \,\|\, \bar{x}, \bar{x}', \bar{\xi})$ as a component of a second order contravariant tensor transported by parallism from (x, x', ξ) to $(\bar{x}, \bar{x}', \bar{\xi})$, we have

$$\delta T^{ij} = dT^{ij} + \Gamma^i_{(h)(k)} T^{hj} dx^k + \Gamma^j_{(h)(k)} T^{ih} dx^k = 0. \quad \ldots(2.22)$$

Using the notation of the differences $\overset{1}{d}, \overset{2}{d}, \overset{3}{d}$ as used earlier, we get

$$\overset{1}{d}T^{ij}(x, x', \xi) = \left[\left(\frac{\partial T^{ij}}{\partial x^k}\right) v^k + \left(\frac{\partial T^{ij}}{\partial x'^k}\right)\left\{\left(\frac{\partial v^k}{\partial x^m}\right) x'^m + \left(\frac{\partial v^k}{\partial \xi^m}\right) \xi'^m\right\}\right.$$

$$\left. + \left(\frac{\partial T^{ij}}{\partial \xi^k}\right)\left\{\left(\frac{\partial v^k}{\partial x^m}\right) \xi^m + \left(\frac{\partial v^k}{\partial \xi^m}\right) d\xi^m\right\}\right] d\tau, \quad \ldots(2.23)$$

$$\overset{2}{d}T^{ij}(x, x', \xi) = \left\{T^{ih}\left(\frac{\partial v^j}{\partial x^h}\right) + T^{hj}\left(\frac{\partial v^i}{\partial x^h}\right)\right\} d\tau \quad \ldots(2.24)$$

and

$$\overset{3}{d}T^{ij}(x, x', \xi) = -\{T^{hj}\Gamma^i_{(h)(k)} + T^{ih}\Gamma^j_{(h)(k)}\} v^k d\tau. \quad \ldots(2.25)$$

Now proceeding on the parallel lines as for the Theorem 2.1 we obtain the Theorem 2.2. Similarly, we have the following.

Theorem 2.3 — Let $T_{ij}(x, x', \xi)$ be the covariant component of second order tensor in $K_n^{(1)}$ under the infinitesimal transformation (2.1), we have

$$\overset{(1,2)}{D}\, T_{ij}(x, x', \xi) = (\nabla_k T_{ij})\, v^k + \left(\frac{\partial T_{ij}}{\partial \xi^k}\right)\left\{(\nabla_h v^k)\, \xi^h + \left(\frac{\partial v^k}{\partial \xi^h}\right) d\xi^h \right.$$

$$\left. - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right) \xi^m + \left(\frac{\partial \xi^k}{\partial x^m}\right) v^m \right\} + \left(\frac{\partial T_{ij}}{\partial x'^k}\right)$$

$$\times \left\{(\nabla_h v^k)\, x'^h - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right) x'^m + \left(\frac{\partial v^k}{\partial \xi^m}\right) \xi'^m \right\}$$

$$+ T_{ih} \left\{(\nabla_j v^h) + \left(\frac{\partial v^h}{\partial \xi^m}\right)\left(\frac{\partial \xi^m}{\partial x^j}\right)\right\}$$

$$+ T_{hj} \left\{(\nabla_i v^h) - \left(\frac{\partial v^h}{\partial \xi^m}\right)\left(\frac{\partial \xi^m}{\partial x^i}\right)\right\}, \qquad \ldots(2.26)$$

$$\overset{(1,3)}{D}\, T_{ij}(x, x', \xi) = (\nabla_k T_{ij})\, v^k + \left(\frac{\partial T_{ij}}{\partial \xi^k}\right)\left\{(\nabla_h v^k)\, \xi^h + \left(\frac{\partial v^k}{\partial \xi^h}\right) d\xi^h \right.$$

$$\left. - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right) \xi^m + \left(\frac{\partial \xi^k}{\partial x^m}\right) v^m \right\} + \left(\frac{\partial T_{ij}}{\partial x'^k}\right)$$

$$\times \left\{(\nabla_h v^k)\, x'^h - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right) x'^m + \left(\frac{\partial v^k}{\partial \xi^m}\right) \xi'^m \right\} \qquad \ldots(2.27)$$

and

$$\overset{(2,3)}{D}\, T_{ij}(x, x', \xi) = - T_{ih} \left\{(\nabla_j v^h) - \left(\frac{\partial v^h}{\partial \xi^k}\right)\left(\frac{\partial \xi^k}{\partial x^j}\right)\right\}$$

$$- T_{hj} \left\{(\nabla_i v^h) - \left(\frac{\partial v^h}{\partial \xi^k}\right)\left(\frac{\partial \xi^k}{\partial x^i}\right)\right\}. \qquad \ldots(2.28)$$

Theorem 2.4 — If $T^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x, x', \xi)$ denotes a mixed tensor of order (r, s) defined in $K_n^{(1)}$ under the infinitesimal transformation (2.1) then the following derivatives hold :

$$\overset{(1,2)}{D}\, T^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x, x', \xi) = (\nabla_k T^{i_1 \cdots i_r}_{j_1 \cdots j_s})\, v^k + \left(\frac{\partial T^{i_1 \cdots i_r}_{j_1 \cdots j_s}}{\partial \xi^k}\right)$$

$$\times \left\{(\nabla_h v^k)\, \xi^h + \left(\frac{\partial v^k}{\partial \xi^h}\right) d\xi^h - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right) \xi^m + \left(\frac{\partial \xi^k}{\partial x^m}\right) v^m \right\}$$

$$+ \left(\frac{\partial T^{i_1 \cdots i_r}_{j_1 \cdots j_s}}{\partial x'^k}\right)\left\{(\nabla_h v^k)\, x'^h - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right) x'^m + \left(\frac{\partial v^k}{\partial \xi^m}\right) \xi'^m \right\}$$

$$- \sum_p T^{i_1 \cdots i_{p-1} k i_{p+1} \cdots i_r}_{j_1 \cdots\cdots\cdots\cdots\cdots j_s} \left\{(\nabla_k v^{i_p}) - \left(\frac{\partial v^{i_p}}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^k}\right)\right\}$$

$$+ \sum_q T^{i_1 \cdots\cdots\cdots\cdots\cdots i_r}_{j_1 \cdots j_{q-1} k j_{q+1} \cdots j_s} \left\{(\nabla_{j_q} v^k) - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^{j_q}}\right)\right\} \ldots(2.29)$$

$$\overset{(1,3)}{D}\, T^{i_1\cdots i_r}_{j_1\cdots j_s}(x, x', \xi) = (\nabla_k T^{i_1\cdots i_r}_{j_1\cdots j_s})\, v^k + \left(\frac{\partial T^{i_1\cdots i_r}_{j_1\cdots j_s}}{\partial \xi^k}\right)$$
$$\times \left\{(\nabla_h v^k)\,\xi^h + \left(\frac{\partial v^k}{\partial \xi^h}\right) d\xi^h - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right)\xi^m + \left(\frac{\partial \xi^k}{\partial x^m}\right) v^m\right\}$$
$$+ \left(\frac{\partial T^{i_1\cdots i_r}_{j_1\cdots j_s}}{\partial x'^k}\right)\left\{(\nabla_h v^k)\, x'^h - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^m}\right) x'^m + \left(\frac{\partial v^k}{\partial \xi^m}\right)\xi'^m\right\} \quad \ldots(2.30)$$

and

$$\overset{(2,3)}{D}\, T^{i_1\cdots i_r}_{j_1\cdots j_s}(x, x', \xi) = \sum_p T^{i_1\cdots i_{p-1} k i_{p+1}\cdots i_r}_{j_1\cdots\cdots\cdots\cdots j_s}\left\{(\nabla_k v^{i_p}) - \left(\frac{\partial v^{i_p}}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^k}\right)\right\}$$
$$- \sum_q T^{i_1\cdots\cdots\cdots\cdots i_r}_{j_1\cdots j_{q-1} k j_{q+1}\cdots j_s}\left\{(\nabla_{j_q} v^k) - \left(\frac{\partial v^k}{\partial \xi^h}\right)\left(\frac{\partial \xi^h}{\partial x^{j_q}}\right)\right\}. \quad \ldots(2.31)$$

PROOF: The Theorem 2.4 is the generalization of the Theorems 2.1, 2.2 and 2.3 and the proof follows on the same lines.

Equations (2.14), (2.15) and (2.16) are the generalizations of the Lie derivative, the covariant derivative and the apparent derivative respectively used by Schouten and Van Kampen (1933) in ordinary Riemannian space.

3. Lie Derivative of the Connection Parameter $\Gamma^i_{(j)(k)}(x, x', \xi)$

As the connection parameter $\Gamma^i_{(j)(k)}(x, x', \xi)$ is not a component of a tensor therefore to obtain the Lie darivative of $\Gamma^i_{(j)(k)}(x, x', \xi)$, we cannot use directly eqn. (2.29). We adopt the following method for obtaining the Lie derivative of the connection parameter.

The law of transformation of $\Gamma^i_{(j)(k)}(x, x', \xi)$ (Rund 1959) from $\bar{x}^i$ to x^i coordinate system under the infinitesimal transformation (2.1) is given by

$$\Gamma^i_{(j)(k)} = \left(\frac{\partial x^i}{\partial \bar{x}^h}\right)\left\{\frac{\partial^2 \bar{x}^h}{\partial x^j \partial x^k} + \bar{\Gamma}^h_{(l)(m)}\left(\frac{\partial \bar{x}^l}{\partial x^j}\right)\left(\frac{\partial \bar{x}^m}{\partial x^k}\right)\right\}. \quad \ldots(3.1)$$

The displaced component of the connection parameter from $\bar{x}^i$ to x^i coordinate system is obtained in view of the eqns. (2.2) and (3.1), we have

$$\overset{2}{d}\Gamma^i_{(j)(k)} = -\left\{\frac{\partial^2 v^i}{\partial x^j \partial x^k} - \left(\frac{\partial v^i}{\partial x^h}\right)\Gamma^h_{(j)(k)} + \left(\frac{\partial v^h}{\partial x^j}\right)\Gamma^i_{(h)(k)} + \left(\frac{\partial v^h}{\partial x^k}\right)\Gamma^i_{(j)(h)}\right\} d\tau. \quad \ldots(3.2)$$

The value of the connection parameter $\Gamma^i_{(j)(k)}(x, x', \xi)$ at $(\bar{x}, \bar{x}', \bar{\xi})$ taking first order approximation in $d\tau$, is given by

$$\Gamma^i_{(j)(k)}(\bar{x}, \bar{x}', \bar{\xi}) = \Gamma^i_{(j)(k)}(x, x', \xi) + \Big[v^h \left(\frac{\partial \Gamma^i_{(j)(k)}}{\partial x^h}\right) + \left(\frac{\partial \Gamma^i_{(j)(k)}}{\partial x'^h}\right)$$
$$\times \left\{\left(\frac{\partial v^h}{\partial \xi^m}\right) x'^m + \left(\frac{\partial v^h}{\partial \xi^m}\right) \xi'^m\right\} + \left(\frac{\partial \Gamma^i_{(j)(k)}}{\partial \xi^h}\right)$$
$$\times \left\{\left(\frac{\partial v^h}{\partial \xi^m}\right) \xi^m + \left(\frac{\partial v^h}{\partial \xi^m}\right) d\xi^m\right\}\Big] d\tau. \qquad \ldots(3.3)$$

From (3.2) and (3.3) we obtain

$$\overset{(1,2)}{D}\Gamma^i_{(j)(k)}(x, x', \xi) = \left(\frac{\partial^2 v^i}{\partial x^j \partial x^k}\right) - \left(\frac{\partial v^i}{\partial x^h}\right)\Gamma^h_{(j)(k)} + \left(\frac{\partial v^h}{\partial x^j}\right)\Gamma^i_{(h)(k)}$$
$$+ \left(\frac{\partial v^h}{\partial x^k}\right)\Gamma^i_{(j)(h)} + v^h \left(\frac{\partial \Gamma^i_{(j)(k)}}{\partial x^h}\right) + \Gamma^i_{(j)(k)(h)}$$
$$\times \left\{\left(\frac{\partial v^h}{\partial x^m}\right) x'^m + \left(\frac{\partial v^h}{\partial \xi^m}\right) \xi'^m\right\} + \left(\frac{\partial \Gamma^i_{(j)(k)}}{\partial \xi^h}\right)$$
$$\times \left\{\left(\frac{\partial v^h}{\partial \xi^m}\right) \xi^m + \left(\frac{\partial v^h}{\partial \xi^m}\right) d\xi^m\right\}. \qquad \ldots(3.4)$$

The operator $\overset{(1,2)}{D} \equiv \underset{v}{\mathcal{L}}$ defines the Lie derivative and eqn. (3.4) then gives the Lie derivative of the connection parameter $\Gamma^i_{(j)(k)}(x, x', \xi)$.

For obtaining Lie derivative of x'^i and x''^i we use eqns. (2.4) and (2.5) and obtain

$$\overset{1}{d}x'^i = \left\{\left(\frac{\partial v^i}{\partial x^h}\right) x'^h + \left(\frac{\partial v^i}{\partial \xi^h}\right) \xi'^h\right\} d\tau,$$

$$\overset{2}{d}x'^i = \left(\frac{\partial v^i}{\partial x^j}\right) x'^j \, d\tau,$$

$$\overset{1}{d}x''^i = \left\{\left(\frac{\partial^2 v^i}{\partial x^j \partial x^k}\right) x'^j x'^k + \left(\frac{\partial^2 v^i}{\partial \xi^j \partial \xi^k}\right) \xi'^j \xi'^k + \left(\frac{\partial v^i}{\partial x^j}\right) x''^j + \left(\frac{\partial v^i}{\partial \xi^j}\right) \xi''^j\right.$$
$$\left. + 2\left(\frac{\partial^2 v^i}{\partial \xi^k \partial x^j}\right) \xi'^k x'^j\right\} d\tau,$$

$$\overset{2}{d}x''^i = \left(\frac{\partial v^i}{\partial x^j}\right) x''^j \, d\tau. \qquad \ldots(3.5)$$

Hence we have

$$\underset{v}{\mathcal{L}} x'^i \equiv \overset{(1,2)}{D} x'^i = \left(\frac{\partial v^i}{\partial \xi^h}\right) \xi'^h \qquad \ldots(3.6)$$

and

$$\underset{v}{\mathcal{L}} x''^i \equiv \overset{(1,2)}{D} x''^i = \left\{\left(\frac{\partial^2 v^i}{\partial x^j \partial x^k}\right) x'^j x'^k + \left(\frac{\partial^2 v^i}{\partial \xi^j \partial \xi^k}\right) \xi'^j \xi'^k + \left(\frac{\partial v^i}{\partial \xi^j}\right) \xi''^j + 2\left(\frac{\partial^2 v^i}{\partial \xi^j \partial x^k}\right) \xi'^j x'^k\right\}. \qquad \ldots(3.7)$$

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