AN ALGORITHM FOR EXTREME POINT MATHEMATICAL PROGRAMMING PROBLEM USING DUALITY RELATIONS

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The present paper, based on duality relations, develops an algorithm for solving an extreme point mathematical programming problem. Dual of a linear programming problem is solved and an additional variable with activity vector normal to the hyperplane representing the objective function is introduced whose cost is so determined that it gives the optimal value of the given extreme point mathematical programming problem. The developed technique is very much different from ranking and cutting plane techniques developed by different authors to solve extreme point mathematical programming problem. To supplement the theory, a numerical example has been solved.

1. INTRODUCTION

An extreme point mathematical programming problem in its most general form is

$$\text{Max} \quad z = CX$$

$$\text{s.t.} \quad AX \leqslant b \tag{P}$$

and X is an extreme point of

$$DX \leqslant d$$

$$X \geqslant 0$$

where C is $1 \times n$, A is $m \times n$, b is $m \times 1$, D is $p \times n$, d is $p \times 1$, 0 and X are $n \times 1$ real matrices.

This was first solved by Kirby *et al.* (1970 ; 1972a, b) Puri and Swarup (1973a, b ; 1975) developed techniques which are improvements over the results of Kirby *et al.* (1970 ; 1972a, b). Any zero-one programming problem can be converted into an extreme point mathematical programming problem and some work has already been done using this approach in special zero-one programming problems namely, Knapsack, Fixed-Charge, Set-Covering, etc., by Murty (1968a, b) Balinski (1961), Arora and Puri (1977) and Arora *et al.* (1978). Extreme point mathematical programming problem is also applicable to Production-Scheduling problem studied by Kirby and Scobey (1970).

In the present paper, we propose to develop a different algorithm for solving an extreme point mathematical programming problem. The present technique depends upon some duality relations which have also been established in this paper. The developed algorithm studies the sensitivity of the optimal solution of dual of a linear programming problem with respect to the cost of an additional variable with known activity vector and determines this cost in such a way that it gives the optimal value of the given problem. The proposed approach is very much different from the ranking techniques developed by Murty (1968a, b) and Kirby *et al.* (1970) and cutting plane technique of Kirby *et al.* (1972a, b).

In the papers by Kirby *et al.* (1972a, b) cuts are introduced which generate alternate solutions of $DX \leqslant d$, $X \geqslant 0$ which are investigated inspite of their known character that they cannot be optimal solutions of the original extreme point linear programming problem. Study of these alternate solutions unnecessarily makes the procedure combursome and time consuming. In a paper by Kirby *et al.* (1970), various extreme points of $DX \leqslant d, X \geqslant 0$ are ranked by enumeration technique where at each stage, we have to consider a new basis for finding the next best extreme point solution., In this approach, procedure starts from a point which is quite far away from the optimal solution of extreme point linear programming problem. In the proposed approach, these weaknesses are overcome in the sense that

(i) no cuts are introduced

(ii) no unwanted alternate solutions are generated

(iii) the process starts from a point nearer to the optimal solution

(iv) unlike Kirby *et al.* (1970), the various extreme point solutions are derivable from the simplex table obtained on introducing the activity vector corresponding to the new variable in the basis.

The developed technique is very useful when the number of constraints in $\begin{pmatrix} A \\ D \end{pmatrix} X \leqslant \begin{pmatrix} b \\ d \end{pmatrix}$ is more than the number of variables because in such a case, a smaller basis is to be studied at each stage.

2. Theoretical Development

Let u be the optimal value of problem (P), i.e. u be the maximum value of $z = CX$ at an extreme point of $DX \leqslant d$, $X \geqslant 0$ which is feasible with respect to $AX \leqslant b$.

Now consider the problem :

$$\begin{aligned} \text{Max} \quad & z = CX \\ \text{s.t.} \quad & AX \leqslant b \\ & DX \leqslant d \\ & CX \leqslant u \\ & X \geqslant 0 \end{aligned} \tag{P.1}$$

Clearly, optimal value of (P) is same as that of (P.1).

Theorem 1 — Every feasible solution of (P) is a basic feasible solution of (P.1).

PROOF : Let X be any feasible solution of (P).

Therefore, X is an extreme point of

$$\begin{aligned} DX &\leqslant d \\ X &\geqslant 0 \end{aligned}$$

and satisfies $AX \leqslant b$.

Since u is assumed to be the optimal value of (P) i.e. $CX = u$, clearly X is a feasible solution of (P.1).

If possible, let X be not a basic feasible solution of (P.1).

Therefore, there exists two distinct feasible solutions X_1, X_2 (say) of (P.1) such that

$$X = \lambda X_1 + (1 - \lambda) X_2, \; 0 < \lambda < 1.$$

Since X_1, X_2 are feasible solutions of (P.1)

$$\begin{aligned} DX_1 &\leqslant d \\ DX_2 &\leqslant d \\ X_1, X_2 &\geqslant 0 \\ DX &= \lambda DX_1 + (1 - \lambda) DX_2 \\ &\leqslant \lambda d + (1 - \lambda) d \\ &= d. \end{aligned}$$

Thus X is not an extreme point of

$$\begin{aligned} DX &\leqslant d \\ X &\geqslant 0 \end{aligned}$$

which contradicts the fact that X is an extreme point of

$$DX \leqslant d$$
$$X \geqslant 0.$$

Hence the theorem is proved.

Corollary — Optimal feasible solution of (P) is an optimal basic feasible solution of (P.1)

Dual of (P.1) is

$$\text{Min } Z = b'w_1 + d'w_2 + uw_0$$
$$\text{s.t.} \qquad A'w_1 + D'w_2 + C'w_0 \geqslant C' \tag{D.1}$$
$$w_1, w_2, w_0 \geqslant 0$$

where w_1 is $m \times 1$, w_2 is $p \times 1$ real matrices and w_0 is a real scalar.

Theorem 2 — If X is any feasible solution of (P) and (w_1, w_2, w_0) is any feasible solution of (D.1), then

$$z \leqslant Z$$

i.e. $$CX \leqslant b'w_1 + d'w_2 + uw_0.$$

PROOF : Since X is a feasible solution of (P), it is also a feasible solution of (P.1). As (P.1) and (D.1) are dual of each other, the rest of the proof follows from duality theory in linear programming.

Theorem 3 — If $\hat{X}$ is a feasible solution to (P) and $\hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_0)$ is a feasible solution of (D.1) such that

$$\hat{z} = \hat{Z}$$

i.e. $$C\hat{X} = b'\hat{w}_1 + d'\hat{w}_2 + u\hat{w}_0$$

then $\hat{X}$ is optimal solution of (P) and $\hat{w}$ is optimal solution of (D.1).

PROOF : Let X be any feasible solution of (P)

By theorem 2,

$$CX \leqslant b'\hat{w}_1 + d'\hat{w}_2 + u\hat{w}_0$$

i.e. $$CX \leqslant C\hat{X}$$

and Hence $\hat{X}$ is optimal solution to (P).

Further, if $w = (w_1, w_2, w_0)$ be any feasible solution of (D.1), then by Theorem 2

$$C\hat{X} \leqslant b'w_1 + d'w_2 + uw_0$$

i.e. $$b'\hat{w}_1 + d'\hat{w}_2 + u\hat{w}_0 \leqslant b'w_1 + d'w_2 + uw_0$$

which implies that $\hat{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_0)$ is an optimal solution of (D.1)

Hence the theorem is proved.

A relaxed form of (P.1) is

$$\begin{aligned} &\text{Max } z = CX \\ &\text{s.t. } AX \leqslant b \\ &\quad DX \leqslant d \\ &\quad X \geqslant 0 \end{aligned} \tag{P.2}$$

clearly, optimal value of (P.1) $\leqslant$ optimal value of (P.2) $= u_1$ (say)

$$\text{i.e.} \quad u \leqslant u_1. \qquad \ldots(1)$$

This gives an upper bound for u.

Theorem 4 — Optimal solution of (P) is an extreme point solution of (P.2).

PROOF : See Kirby *et al.* (1970, 1972a, 1972b).

Dual of (P.2) is

$$\begin{aligned} &\text{Min } v = b'w_1 + d'w_2 \\ &\text{s.t.} \quad A'w_1 + D'w_2 \geqslant c' \\ &\quad\quad w_1, w_2 \geqslant 0. \end{aligned} \tag{D.2}$$

From duality theory,

Optimal value of (D.2) = optimal value of (P.2) $= u_1$. (D.1) differs from (D.2) in the sense that there is an extra variable w_0 with cost u (to be determined) and activity vector c'.

After getting an optimal solution of (D.2), the additional variable w_0 with activity c' and cost u is introduced. This gives initial basic feasible solution for (D.1), with initial basis same as optimal basis of (D.2), say B.

For w_0,

$$y_j = B^{-1}c' = w_B,$$

where w_B is the vector of optimal basic variables of (D.2).

Also for w_0, the relative cost, $f_j - z_j$ where f_j is jth component of $f = (b', w', u)$ is given by

$$
\begin{aligned}
z_j - f_j &= f_B y_j - u \qquad \text{where } f_B \text{ is the vector of prices corresponding to basic variables.}\\
&= f_B w_B - u\\
&= u_1 - u\\
&\geqslant 0 \qquad \text{(from (1)).}
\end{aligned}
$$

w_0 is entered into the basis. Also, because y_j for w_0 is w_B ; w_0 enters the basis at level 1 rendering all other basic variables zero.

3. Procedure

Let

X_j = set of all jth best extreme point solutions of (P.2)

E_1 = set of all bases of (D.1) adjacent to optimal bases of (D.2) from which they are derived by finding u_{min} so that each satisfies optimality criterion

B_j = set of all bases of (D.1) corresponding to X_j

= set of all bases of (D.1) corresponding to

$$u_j = \underset{H_{j-1}}{\text{Max}}\,[\text{Min } u], \quad \text{for} \quad j \geqslant 3$$

E_j = set of all bases of (D.1) with $u \leqslant u_j$, adjacent to B_j from which they are derived by finding u_{min} so that each satisfies optimality criterion. $(j \geqslant 2)$

$S_1 = \{X : X \text{ is an extreme point of } DX \leqslant d,\ X \geqslant 0\}$

$H_1 = E_1$

$$H_j = \bigcup_{i=1}^{j} E_i \setminus \bigcup_{i=2}^{j} B_i \quad \text{for} \quad j \geqslant 2.$$

From optimal solution of (D.2) obtain the set X_1 of all optimal extreme point solutions of (P.2). Test to see if $X_1 \cap S_1 \neq \phi$, which if so implies by Theorem 4, that every $X \in X_1 \cap S_1$ is an optimal solution of (P). But if $X_1 \cap S_1 = \phi$, proceed to find X_2.

In the initial simplex table of (D.1) we have for w_0, $y_j = w_B$ so any of the basic variables from w_B can be replaced by w_0 giving us various solutions of (D.1). Minimum values of u which ensure optimality for (D.1) for each of these solutions are found. All these solutions of (D.1) form the set E_1 and correspond to basic feasible solutions of (P.2) adjacent to X_1. Further, the set B_2 of all bases of (D.1) corresponding to $u_2 = \max_{E_1} [\text{Min } u]$ yields optimal solution of (D.1) and so correspondingly, optimal solution of (P.1) for $u = u_2$. Clearly B_2 corresponds to the set X_2 of all the 2nd best extreme point solutions of (P.2). Test again to see if $X_2 \cap S_1 \neq \phi$, which if so implies that every $X \in X_2 \cap S_1$ is an optimal solution of (P).

If $X_2 \cap S_1 = \phi$, proceed further to find E_2, the set of all bases of (D.1) with $u \leqslant u_2$, adjacent to B_2. Determine the set $H_2 = E_1 \cup E_2 \setminus B_2$ and pick up the bases from H_2 which correspond to $u_3 = \text{Max}_{H_2} [\text{Min } u]$. This gives B_3, the set of all bases of (D.1) which correspond to the set X_3 of all 3rd best extreme point solutions of (P.2). Again X_3 is tested to see if $X_3 \cap S_1 \neq \phi$, which if so implies every $X \in X_3 \cap S_1$ is an optimal solution of (P).

If $X_3 \cap S_1 = \phi$, proceed to find X_4 and so on. Continue till at some stage say kth we either get $X_k \cap S_1 \neq \phi$, in which case every $X \in X_k \cap S_1$ is an optimal solution of (P) ; or $\{X_k \neq \phi, X_k \cap S_1 = \phi, H_k = \phi\}$ which implies no solution for (P).

Convergence of the process is guaranteed by the fact that there are only a finite number of extreme points of (P.2) and no extreme point is ever repeated as

$$u_1 > u_2 > u_3 > \ldots > u_j > \ldots$$

i.e. because value of the objective function is reduced at every stage.

4. Algorithm

Step 1 — Solve (D.2) and obtain the set X_1 of all optimal solutions of its primal (P.2).

If $X_1 \cap S_1 \neq \phi$, then every $X \in X_1 \cap S_1$ is an optimal solution of (P) and so the process terminates.

If $X_1 \cap S_1 = \phi$ go to step 2.

Step 2 — Introducing w_0 in the optimal simplex table of (D.2) determine the set E_1.

Then X_2 is obtained corresponding to $u_2 = \text{Max}_{E_1} [\text{Min } u]$.

If $X_2 \cap S_1 \neq \phi$, then every $X \in X_2 \cap S_1$ is an optimal solution of (P) and so the process terminates.

If $X_2 \cap S_1 = \phi$ go to step 3.

Step 3 — Find $X_i (i \geqslant 3)$ corresponding to $u_i = \text{Max}_{H_{i-1}} [\text{Min } u]$.

If $X_i \cap S_1 \neq \phi$, then every $X \in X_i \cap S_1$ is an optimal solution of (P) and so the process terminates.

If $X_i \cap S_1 = \phi$ go to step 4.

Step 4 — Repeat step 3, for next higher value of i, i.e. for $i = i + 1$.

5. Remarks

(1) Problem (P.2) is always assumed to be bounded. Because, if it is not so, we can make it bounded by introducing the constraint $CX \leqslant M$, where M is sufficiently large positive number. In this case, the role of u_1 is played by M.

(2) Our basic aim was to find dual of extreme point programming problem (P). We have yet not succeeded in that, however, in quest of its search we have developed the technique presented in this paper for solving the problem (P). Problem (D.1) with final value of u which gives optimal solution of (P) will be dual of (P) for which duality relations will be satisfied by virtue of theorems 1, 2 and 3. How to obtain dual of (P) without knowing final u is a problem still unsolved.

6. Example

$$\begin{aligned} \text{Max } z &= x_1 + 3x_2 \\ \text{s.t.} \quad & -3x_1 + 2x_2 \leqslant 2 \\ & 2x_2 \leqslant 5 \end{aligned} \tag{P}$$

and (x_1, x_2) is an extreme point of

$$\begin{aligned} -x_1 + 2x_2 &\leqslant 4 \\ 2x_1 + 5x_2 &\leqslant 19 \\ 2x_1 - x_2 &\leqslant 7 \\ x_1, x_2 &\geqslant 0. \end{aligned}$$

Solution

Problem (P.2) becomes

$$\begin{aligned} \text{Max } z &= x_1 + 3x_2 \\ \text{s.t.} \quad & -3x_1 + 2x_2 \leqslant 2 \\ & 2x_2 \leqslant 5 \\ & -x_1 + 2x_2 \leqslant 4 \\ & 2x_1 + 5x_2 \leqslant 19 \\ & 2x_1 - x_2 \leqslant 7 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

and problem (D.2) is given by

$$\begin{aligned} \text{Min } v &= 2w_1 + 5w_2 + 4w_3 + 19w_4 + 7w_5 \\ \text{s.t.} \quad & -3w_1 - w_3 + 2w_4 + 2w_5 \geqslant 1 \\ & 2w_1 + 2w_2 + 2w_3 + 5w_4 - w_5 \geqslant 3 \\ & w_i \geqslant 0,\ i = 1, 2, \ldots, 5. \end{aligned}$$

Add surplus variables w_6, w_7 and artificial variables w_{a_1}, w_{a_2} (with cost M, where M is sufficiently large positive number) to the constraints of (D.2) and solve.

Optimal solution of (D.2) is

$$X_1 = \{(x_1 = 13/4,\ x_2 = 5/2)\}$$
$$u_1 = 43/4$$
also $X_1 \cap S_1 = \phi$.

Introducing the additional variable w_0 with activity vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and cost u and inserting a_0 in the basis, we get Tables I and II.

Minimum value of u for which the solution (w_0, w_2) in Table I satisfies optimality criterion is 17/2.

Minimum value of u for which the solution (w_4, w_0) in Table II satisfies optimality criterion is 21/2.

$$E_1 = \{(a_0, a_2), (a_4, a_0)\}$$

$$u_2 = \operatorname*{Max}_{E_1} [\text{Min } u] = \text{Max } [17/2, 21/2] = 21/2.$$

$$B_2 = \{(a_4, a_0)\}$$

Corresponding to basis B_2, the set X_2 of second best extreme point solutions of (P.2) is given by

$$X_2 = \{(x_1 = 9/2,\ x_2 = 2)\}$$
As $X_2 \cap S_1 \neq \phi$.

Hence $x_1 = 9/2$, $x_2 = 2$ give optimal solution of (P) with optimum value of the objective function equal to 21/2.

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TABLE I

f_B	Vectors in the basis	W_B	a_1	a_2	a_3	a_4	a_5	a_6	a_7	q_1	q_2	a_0
u	a_0	1	-3	0	-1	2	2	-1	0	1	0	1
5	a_2	0	11/2	1	5/2	$-1/2$	$-7/2$	3/2	$-1/2$	$-3/2$	1/2	0
	$z_j - f_j$	u	$(-3u + 51/2)$	0	$(-u + 17/2)$	$(2u - 43/2)$	$(2u - 49/2)$	$(-4u + 15)$	$-5/2$	$(u - 15/2 - M)$	$(5/2 - M)$	0

TABLE II

f_B	Vectors in the basis	w_B	a_1	a_2	a_3	a_4	a_5	a_6	a_7	q_1	q_2	a_3
19	a_4	0	-11	-2	-5	1	7	-3	1	3	-1	0
u	a_0	1	19	4	9	0	-12	5	-2	-5	2	1
	$z_j - f_j$	u	$(19u - 211)$	$(4u - 43)$	$(9u - 99)$	0	$(-12u + 126)$	$(5u - 57)$	$(-2u + 19)$	$(-5u + 57 - M)$	$(2u - 19 - M)$	0

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