APPLICATION OF GENERALIZED HYPERGEOMETRIC FUNCTIONS TO GENERALIZED BIRTH AND DEATH PROCESSES

by J. N. KAPUR, *Department of Mathematics, Indian Institute of Technology, Kanpur* 208016

(*Received* 6 *March* 1978)

In the present paper, we (i) enumerate the number of possible relations between probabilities of ultimate extinction of contiguous birth and death processes, (ii) state thirteen of these relations, (iii) establish the existence of linear relationship between a specified number of consecutive moments of the probability distribution of ultimate population size and (iv) find these relationships for three specified birth and death processes.

1. INTRODUCTION

In a recent paper (Kapur 1978), we have shown that if p_n is the steady-state probability of there being n persons in the system when the probabilities of birth and death in the time interval $(t, t + \Delta t)$ are $\lambda_n \Delta t + O(\Delta t)$ and $\mu_n \Delta t + O(\Delta t)$ respectively, then the following cases arise :

Case I — If
$$\left.\begin{aligned} \lambda_n &= (na_1 + b_1)(na_2 + b_2) \ldots (na_p + b_p) \\ \mu_n &= (nc_1 + d_1)(nc_2 + d_2) \ldots (nc_q + d_q) \end{aligned}\right] \qquad \ldots(1)$$

then

$$p_n = p_0 \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_p)_n}{(\gamma_1)_n (\gamma_2)_n \ldots (\gamma_q)_n}, \qquad \ldots(2)$$

where

$$(\alpha)_n = \alpha(\alpha + 1) \ldots (\alpha + n - 1), (n \geqslant 1); (\alpha)_0 = 1 \qquad \ldots(3)$$

$$\alpha_i = \frac{b_i}{c_i}, \ \gamma_j = \frac{d_j}{c_j} + 1, \ (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q) \qquad \ldots(4)$$

$$x = \frac{a_1 a_2 \ldots a_p}{c_1 c_2 \ldots c_q} \qquad \ldots(5)$$

$$p_0 = \left\{ {}_{p+1}F_q \left[\begin{matrix} 1 \ \alpha_1 \alpha_2 \ldots \alpha_p \\ \gamma_1 \gamma_2 \ldots \gamma_q \end{matrix} ; x \right] \right\}^{-1} \qquad \ldots(6)$$

$$\mu'_{(r)} = \lfloor r \, \frac{(\alpha_1)_r (\alpha_2)_r \dots (\alpha_p)_r}{(\gamma_1)_r (\gamma_2)_r \dots (\gamma_q)_r} \, x^r \, \frac{{}_{p+1}F_q \left[\begin{matrix} 1+r & \alpha_1+r \dots \alpha_p+r \\ & \gamma_1+r \dots \gamma_q+r \end{matrix} ; x \right]}{{}_{p+q}F_q \left[\begin{matrix} 1 & \alpha_1 \dots \alpha_p \\ & \gamma_1 \dots \gamma_q \end{matrix} ; x \right]}. \quad \dots(7)$$

In particular if $p = q = 1$, i.e., if

$$\lambda_n = na_1 + b_1, \; \mu_n = nc_1 + d_1 \quad \dots(8)$$

we get

$$p_0 = \left\{ {}_2F_1 \left[\begin{matrix} 1 & \alpha_1 \\ & \gamma_1 \end{matrix} ; x \right] \right\}^{-1} \quad \dots(9)$$

$$\mu'_{(r)} = \lfloor r \, \frac{(\alpha_1)_r}{(\gamma_1)_r} \, x^r \, \frac{{}_2F_1 \left[\begin{matrix} 1+r & \alpha_1+r \\ & \gamma_1+r \end{matrix} ; x \right]}{{}_2F_1 \left[\begin{matrix} 1 & \alpha_1 \\ & \gamma_1 \end{matrix} ; x \right]}. \quad \dots(10)$$

Case II — If

$$\left. \begin{aligned} \lambda_n &= (na_1 + b_1)(na_2 + b_2) \dots (na_p + b_p) \\ \mu_n &= n(nc_1 + d_1)(nc_2 + d_2) \dots (nc_q + d_q) \end{aligned} \right] \quad \dots(11)$$

then

$$p_n = p_0 \frac{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_p)_n}{(\gamma_1)_n (\gamma_2)_n \dots (\gamma_q)_n} \frac{x^n}{\lfloor n} \quad \dots(12)$$

$$p_0 = \left\{ {}_pF_q \left[\begin{matrix} \alpha_1 \alpha_2 \dots \alpha_p \\ \gamma_1 \gamma_2 \dots \gamma_q \end{matrix} ; x \right] \right\}^{-1} \quad \dots(13)$$

$$\mu'_{(r)} = \frac{(\alpha_1)_r (\alpha_2)_r \dots (\alpha_p)_r}{(\gamma_1)_r (\gamma_2)_r \dots (\gamma_q)_r} \, x^r \, \frac{{}_pF_q \left[\begin{matrix} \alpha_1+r \dots \alpha_p+r \\ \gamma_1+r \dots \gamma_q+r \end{matrix} ; x \right]}{{}_pF_q \left[\begin{matrix} \alpha_1 \dots \alpha_p \\ \gamma_1 \dots \gamma_q \end{matrix} ; x \right]}. \quad \dots(14)$$

As a particular case, if

$$\lambda_n = (na_1 + b_1), \; \mu_n = n(nc_1 + d_1) \quad \dots(15)$$

$$p_0 = \left\{ {}_1F_1 \left[\begin{matrix} \alpha_1 \\ ; x \\ \gamma_1 \end{matrix} \right] \right\}^{-1} \quad ...(16)$$

$$\mu'_{(r)} = \frac{(\alpha_1)_r}{(\gamma_1)_r} x^r \frac{{}_1F_1 \left[\begin{matrix} \alpha_1 + r \\ ; x \\ \gamma_1 + r \end{matrix} \right]}{{}_1F_1 \left[\begin{matrix} \alpha_1 \\ ; x \\ \gamma_1 \end{matrix} \right]}. \quad ...(17)$$

The general birth and death process given above is characterised by $2p + 2q$ parameters a_i, b_i, c_j, d_j ($i = 1, 2, ..., p$; $j = 1, 2, ..., q$). Another birth and death process is said be contiguous to it if all the parameters are the same except that either b_i is replaced by $b_i + a_i$ or $b_i - a_i$ or d_j is replaced by $d_j + c_j$ or $d_j - c_j$ for one particular value of i or j.

Since contiguous hypergeometric functions are related, contiguous birth and death processes are also related. In particular, the probabilities of ultimate extinction are related for such processes.

Kapur (1978) stated fifteen such relations between the probabilities of extinction of contiguous processes corresponding to

$$\lambda_n = (na_1 + b_1)(na_2 + b_2), \ \mu_n = n(nc_1 + d_1). \quad ...(18)$$

Also stated are four relations for the class of birth and death processes given by (11) and one for the class given by (8). We give here some more relations in the next section.

2. More Relations Between Probabilities of Ultimate Extinction of Contiguous Birth and Death Processes

2.1. Let

$$\lambda_n = na + b, \ \mu_n = nc + d \quad ...(19)$$

then if $\Gamma_0[a, b, c, d]$ denotes the reciprocal of the probability of ultimate extinction for this process, we get the following six relations by making use of the relations between contiguous functions given in Magnus *et al.* (1966)

$$\left(\frac{d}{c} - \frac{b}{a} + 1\right) \Gamma_0[a, b - a, c, d] + \left[2\frac{b}{a} - \frac{d}{c} - 1 - \left(\frac{b}{a} - 1\right)\frac{a}{c}\right] \Gamma_0[a, b, c, d] + \frac{b}{a}\left(\frac{a}{c} - 1\right) \Gamma_0[a, b + a, c, d] = 0 \quad ...(20)$$

$$\frac{d}{c}\left(\frac{d}{c}+1\right)\left(\frac{a}{c}-1\right)\Gamma_0\,[a, b, c, d-c]+\left(\frac{d}{c}-1\right)$$
$$\times\left[\frac{d}{c}-\left(\frac{2d}{c}-\frac{b}{a}\right)\frac{a}{c}\right]\Gamma_0\,[a, b, c, d]$$
$$+\frac{d}{c}\left(\frac{d}{c}+1-\frac{b}{a}\right)\frac{a}{c}\,\Gamma_0\,[a, b, c, d+c]=0 \qquad \ldots(21)$$

$$\left(\frac{d}{c}+1\right)\left(\frac{b}{a}-\frac{ad}{c^2}\right)\Gamma_0\,[a, b, c, d]$$
$$-\frac{b}{a}\left(\frac{d}{c}+1\right)\left(1-\frac{a}{c}\right)\Gamma_0\,[a, b+a, c, d]$$
$$+\frac{ad}{c^2}\left(\frac{d}{c}-\frac{b}{a}\right)\Gamma_0\,[a, b, c, d+c]=0 \qquad \ldots(22)$$

$$-\frac{d}{c}\,\Gamma_0\,[a, b, c, d-c]+\left(\frac{d}{c}-\frac{b}{a}\right)\Gamma_0\,[a, b, c, d]$$
$$+\frac{b}{a}\,\Gamma_0\,[a, b+a, c, d]=0 \qquad \ldots(23)$$

$$-\left(\frac{d}{c}+1\right)\Gamma_0\,[a, b-a, c, d]+\left(\frac{d}{c}+1\right)\left(1-\frac{a}{c}\right)\Gamma_0\,[a, b, c, d]$$
$$+\frac{ad}{c^2}\,\Gamma_0\,[a, b, c, d+c]=0 \qquad \ldots(24)$$

$$\left(\frac{d}{c}-\frac{b}{a}+1\right)\Gamma_0\,[a, b-a, c, d]$$
$$-\frac{d}{c}\left(1-\frac{a}{c}\right)\Gamma_0\,[a, b, c, d-c]$$
$$+\left[\frac{b}{a}-1-\left(\frac{d}{c}-1\right)\frac{a}{c}\right]\Gamma_0\,[a, b, c, d]=0. \qquad \ldots(25)$$

2.2. Let

$$\lambda_n = na+b, \;\mu_n = n(nc+d). \qquad \ldots(26)$$

We gave four relations for this case earlier. We give seven more below :

$$\frac{d}{c}\,\Gamma_0\,[a, b, c, d-c]+\left(\frac{b}{a}-\frac{d}{c}\right)\Gamma_0\,[a, b, c, d]$$
$$-\frac{b}{a}\,\Gamma_0\,[a, b+a, c, d]=0 \qquad \ldots(27)$$

$$\left(\frac{d}{c}+1\right)\Gamma_0[a,b,c,d]-\left(\frac{d}{c}+1\right)\Gamma_0[a,b-a,c,d]$$
$$-\frac{a}{c}\Gamma_0[a,b,c,d+c]=0 \qquad ...(28)$$

$$\left(\frac{d}{c}+1\right)\left(\frac{b}{a}+\frac{a}{c}\right)\Gamma_0[a,b,c,d]$$
$$-\left(\frac{d}{c}-\frac{b}{a}+1\right)\frac{a}{c}\Gamma_0[a,b,c,d+c]$$
$$-\frac{b}{a}\left(\frac{d}{c}+1\right)\Gamma_0[a,b+a,c,d]=0 \qquad ...(29)$$

$$\left(\frac{d}{c}-\frac{b}{a}+1\right)\Gamma_0[a,b-a,c,d]-\frac{d}{c}\Gamma_0[a,b,c,d-c]$$
$$+\left(\frac{b}{a}+\frac{a}{c}-1\right)\Gamma_0[a,b,c,d]=0 \qquad ...(30)$$

$$\left(\frac{d}{c}+1\right)\left(\frac{d}{c}-\frac{b}{a}+1\right)\Gamma_0[a,b-a,c,d]$$
$$-\left(\frac{d}{c}+1\right)\left(\frac{d}{c}-\frac{b}{a}-\frac{a}{c}+1\right)\Gamma_0[a,b,c,d]$$
$$-\frac{b}{c}\Gamma_0[a,b+a,c,c+d]=0 \qquad ...(31)$$

$$\frac{d}{c}\left(\frac{d}{c}+1\right)\Gamma_0[a,b,c,d-c]-\frac{d}{c}\left(\frac{d}{c}+1\right)\Gamma_0[a,b,c,d]$$
$$-\frac{b}{c}\Gamma_0[a,b+a,c,d+c]=0 \qquad ...(32)$$

$$\frac{d}{c}\left(\frac{d}{c}+1\right)\Gamma_0[a,b-a,c,d-c]$$
$$+\left(\frac{d}{c}+1\right)\left(\frac{a}{c}-\frac{d}{c}\right)\Gamma_0[a,b,c,d]$$
$$-\frac{b}{c}\Gamma_0[a,b+a,c,d+c]=0. \qquad ...(33)$$

2.3. Kapur (1978) gave twenty relations and we have given thirteen more above. In fact some of the birth and death processes involved above are doubly contiguous since they involve changes in two arguments.

In the next section, we investigate the total number of such relations possible in each case.

3. Number of Possible Relations Between Contiguous Birth and Death Processes

Rainville (1945, 1960) has shown that among the $2p + 2q$ functions contiguous to

$$ {}_pF_q\left[\begin{matrix}\alpha_1\alpha_2 \ldots \alpha_p \\ \gamma_1\gamma_2 \cdots \gamma_q\end{matrix} ; x\right] \quad \ldots(34)$$

and the function itself, there is a set of $2p + q$ linearly independent relations and he has explicitly obtained these. From these relations we can eliminate $2p + q - 1$ functions and get a relation between the function (34) and $(q + 1)$ of its contiguous functions. We can choose these functions in

$$ {}^{2p+2q}C_{q+1} = \frac{(2p + 2q)!}{(q + 1)!\,(2p + q - 1)!} \quad \ldots(35)$$

different ways and as such we can get as many relations, each involving (34) and $(q + 1)$ of the contiguous functions. Since to every such relation, we get a linear relation between the reciprocals of the probabilities of ultimate extinction of contiguous birth and death processes, the number of such relations for the ultimate probabilities of extinction given by (13) is given by (35).

For the generalized hypergeometric function in (6) also, there are $2p + 2q$ contiguous functions since the argument 1 has not to be changed and we have still to choose $q + 1$ functions out of these, so that the number of relations is still given by (35).

We thus get the following table :

Birth and death rates		Number of relations
$\lambda_n = (na_1 + b_1)$,	$\mu_n = (nc_1 + d_1)$	6
$\lambda_n = (na_1 + b_1)$,	$\mu_n = (nc_1 + d_1)(nc_2 + d_2)$	20
$\lambda_n = (na_1 + b_1)$,	$\mu_n = n(nc_1 + d_1)$	6
$\lambda_n = (na_1 + b_1)(na_2 + b_2)$,	$\mu_n = n(nc_1 + d_1)$	15
$\lambda_n = (na_1 + b_1)(na_2 + b_2)$,	$\mu_n = (nc_1 + d_1)(nc_2 + d_2)$	56
$\lambda_n = (na_1 + b_1)(na_2 + b_2)$,	$\mu_n = n(nc_1 + d_1)(nc_2 + d_2)$	56
$\lambda_n = (na_1 + b_1)(na_2 + b_2)(na_3 + b_3)$,	$\mu_n = n(nc_1 + d_1)(nc_2 + d_2)$	120

4. Linear Relation Between Consecutive Moments

Since there is a linear relation between ${}_pF_q$ and any $q+1$ of its contiguous functions, we can express every contiguous function as a linear combination of ${}_pF_q$ and q fixed contiguous function.

By repeating this process, we can express

$$ {}_pF_q\left[\begin{matrix}\alpha_1+l_1 & \alpha_2+l_2 \ldots \alpha_p+l_p \\ & ;x \\ \gamma_1+m_1 & \gamma_2+m_2 \ldots \gamma_q+m_q\end{matrix}\right] \qquad \ldots(36)$$

where l's and m's are integers and none of γ_j+m_j are zero, as a linear combination of ${}_pF_q$ and q fixed contiguous function; the coefficients will be rational functions of the parameters.

In particular we can express each of

$$ {}_pF_q\left[\begin{matrix}\alpha_1+r & \alpha_2+r \ldots \alpha_p+r \\ & ;x \\ \gamma_1+r & \gamma_2+r \ldots \gamma_q+r\end{matrix}\right], (r=1, 2, \ldots, q+1) \qquad \ldots(37)$$

as a linear combination of ${}_pF_q$ and these q functions. Eliminating these q contiguous functions, we get a linear relation of the form

$$ \sum_{r=0}^{q+1} c_r\, {}_pF_q\left[\begin{matrix}\alpha_1+r & \alpha_2+r \ldots \alpha_p+r \\ & ;x \\ \gamma_1+r & \gamma_2+r \ldots \gamma_q+r\end{matrix}\right]=0 \qquad \ldots(38)$$

where c's are rational functions of parameters. This gives

$$ \sum_{r=0}^{q+1} c_r \frac{{}_pF_q\left[\begin{matrix}\alpha_1+r & \alpha_2+r \ldots \alpha_p+r \\ & ;x \\ \gamma_1+r & \gamma_2+r \ldots \gamma_q+r\end{matrix}\right]}{{}_pF_q\left[\begin{matrix}\alpha_1\alpha_2 \ldots \alpha_p \\ ;x \\ \gamma_1\gamma_2 \cdots \gamma_q\end{matrix}\right]}=0. \qquad \ldots(39)$$

Using (14) we get

$$ \sum_{r=0}^{q+1} d_r\, \mu'_{(r)}=0 \qquad \ldots(40)$$

where d's are again rational functions of parameters. We thus get a linear relation between $q+2$ factorial moments $\mu'_{(0)}, \mu'_{(1)}, \ldots, \mu'_{(q+1)}$. This would give a linear relation between the $(r+2)$ moments $\mu'_0, \mu'_1, \ldots, \mu'_{q+1}$ about the origin of the type

$$\sum_{r=0}^{q+1} e_r \mu'_r = 0. \qquad \ldots(41)$$

Equation (38) can be replaced by

$$\sum_{r=k}^{q+1+k} f_r \, {}_pF_q \left[\begin{matrix} \alpha_1 + r \;\; \alpha_2 + r \ldots \alpha_p + r \\ \gamma_1 + r \;\; \gamma_2 + r \ldots \gamma_q + r \end{matrix} ; x \right] = 0. \qquad \ldots(42)$$

By dividing (42) by ${}_pF_q \left[\begin{matrix} \alpha_1 \alpha_2 \ldots \alpha_p \\ \gamma_1 \gamma_2 \cdots \gamma_q \end{matrix} ; x \right]$ and using (14), we get a linear relation of the form

$$\sum_{r=k}^{q+1+k} g_r \mu'_{(r)} = 0 \text{ or } \sum_{r=k}^{q+1+k} h_r \mu'_r = 0. \qquad \ldots(43)$$

This shows that any $q + 2$ consecutive moments about a fixed point are linearly related.

We may note that the number of consecutive moments involved depends on q but not on p. We also note that $q + 2$ consecutive moments about the mean are not linearly related.

5. Special Cases

Though in principle, a relation of the form (38) always exists, it is not always easy to find it. We consider three cases below.

Case I — Let

$$\lambda_n = na_1 + b_1, \; \mu_n = n(nc_1 + d_1) \qquad \ldots(44)$$

Fortunately for this case, the recurrence relation (38) is available in Magnus *et al.* (1966). This is

$$(\gamma + r)(\gamma + r + 1) \, {}_1F_1 \left[\begin{matrix} \alpha + r \\ \gamma + r \end{matrix} ; x \right] + (\gamma + r + 1)(-\gamma - r + x)$$

$$\times \, {}_1F_1 \left[\begin{matrix} \alpha + r + 1 \\ \gamma + r + 1 \end{matrix} ; x \right] - (\alpha + r + 1) \, x \, {}_1F_1 \left[\begin{matrix} \alpha + r + 2 \\ \gamma + r + 2 \end{matrix} ; x \right] = 0.$$

$$\ldots(45)$$

Dividing by ${}_1F_1\left[\begin{matrix}\alpha \\ \gamma\end{matrix}; x\right]$ and using (16) we get

$$(\gamma + r)(\gamma + r + 1)\frac{(\gamma)_r}{(\alpha)_r}\frac{\mu'_{(r)}}{x^r} + (\gamma + r + 1)(-\gamma - r + x)\frac{(\gamma)_{r+1}}{(\alpha)_{r+1}}\frac{1}{x^{r+1}}\mu'_{(r+1)}$$

$$- (\alpha + r + 1)\, x\frac{(\gamma)_{r+2}}{(\alpha)_{r+2}}\frac{1}{x^{r+2}}\mu'_{(r+2)} = 0$$

or $$(\alpha + r)\, x\, \mu'_{(r)} - (\gamma + r - x)\,\mu'_{(r+1)} - \mu'_{(r+2)} = 0. \qquad \ldots(46)$$

In particular this gives

$$\mu'_2 + \mu'_1(\gamma - x - 1) - x = 0 \qquad \ldots(47)$$

$$\mu'_3 + \mu'_2(\gamma - 2 - x)\,\mu'_2 - \mu'_1(\alpha x - 3 - \gamma) = 0. \qquad \ldots(48)$$

If we know ${}_1F_1\left[\begin{matrix}\alpha_1 \\ \gamma_1\end{matrix}; x\right]$ and ${}_1F_1\left[\begin{matrix}\alpha_1 + 1 \\ \gamma_1 + 1\end{matrix}; x\right]$, we can find moments of all orders.

Case II — Let

$$\left.\begin{aligned}\lambda_n &= (na_1 + b_1)(na_2 + b_2) = a_1a_2(n + \alpha_1)(n + \alpha_2) \\ \mu_r &= n(nc_1 + d_1) = nc_1(n + \gamma_1 - 1).\end{aligned}\right\} \qquad \ldots(49)$$

Using Gauss's relations between contiguous hypergeometric functions eight times and after a great deal of simplification, we get

$$(1 - x)\,[(\alpha_2 + r)(\gamma_1 + r + 1) - (\alpha_1 + r + 1)(\alpha_2 + r + 1)\, x]$$

$$\times\, {}_2F_1\left[\begin{matrix}\alpha_1 + r + 2 \quad \alpha_2 + r + 2 \\ \gamma_1 + r + 2\end{matrix}; x\right]$$

$$+ (\gamma_1 + r + 1)(\gamma_1 - \alpha_2 - (\alpha_1 + r + 1))\, {}_2F_1\left[\begin{matrix}\alpha_1 + r + 1 \quad \alpha_2 + r + 1 \\ \gamma_1 + r + 1\end{matrix}; x\right]$$

$$- (\gamma_1 + r)(\gamma_1 + r + 1)\, {}_2F_1\left[\begin{matrix}\alpha_1 + r \quad \alpha_2 + r \\ \gamma_1 + r\end{matrix}; x\right] = 0. \qquad \ldots(50)$$

Dividing by ${}_2F_1\left[\begin{matrix}\alpha_1\alpha_2 \\ \gamma_1\end{matrix}; x\right]$, using (14) and simplifying, we get

$$(1-x)\,[(\alpha_2+r)(\gamma_1+r+1)-(\alpha_1+r+1)(\alpha_2+r+1)\,x]\,\mu'_{(r+2)}$$
$$+\,[\gamma_1-\alpha_2-(\alpha_1+r+1)\,x]\,(\alpha_1+r+1)\,(\alpha_2+r+1)\,x\,\mu'_{(r+1)}$$
$$-\,(\alpha_1+r)\,(\alpha_1+r+1)\,(\alpha_2+r)\,(\alpha_2+r+1)\,x^2\,\mu'_{(r)}=0. \quad \ldots(51)$$

Knowing ${}_2F_1\left[\begin{matrix}\alpha_1\alpha_2 \\ & ;x \\ \gamma_1\end{matrix}\right]$ and ${}_2F_1\left[\begin{matrix}\alpha_1+1 \quad \alpha_2+1 \\ & ;x \\ \gamma_1+1\end{matrix}\right]$, we can find moments of all orders.

Case III — Let

$$\lambda_n=na+b=a(n+\alpha),\ \mu_n=nc+d=c(n+\gamma-1). \quad \ldots(52)$$

It can be easily seen that the moments for this process can be deduced from the moments for the process (47) by putting

$$a_1a_2=a,\ c_1=c,\ \alpha_1=\alpha,\ \alpha_2=1,\ \gamma_1=\gamma \quad \ldots(53)$$

so that in this case (49) gives

$$(1-x)\,[(\alpha+r)\,(\gamma+r-1)-(r+2)\,(\alpha+r+1)\,x]\,\mu'_{(r+2)}$$
$$+\,[\gamma-1-(\alpha+r+1)\,x]\,(\alpha+r+1)\,(r+2)\,x\,\mu'_{(r+1)}$$
$$-\,(\alpha+r)\,(\alpha+r+1)\,(1+r)\,(r+2)\,x^2\,\mu'_{(r)}=0. \quad \ldots(54)$$

6. Discussion of Case III

This is an important case. We can regard a and c as specific birth and death rates and b and d as immigration and emigration rates. In the case when

$$\lambda_n=a(n+\alpha),\ \mu_n=cn, \quad \ldots(55)$$

we find,

$$\mu_1=\frac{\alpha a}{c-a},\ \mu_2=\frac{\alpha ac}{(c-a)^2}. \quad \ldots(56)$$

The expression for the variance differs from the limiting value of that obtained by Getz (1975, 1976) because of omission of one term in his expression for variance.

It also appears that his expressions for variance in the general case

$$\lambda_n=a(n+\alpha),\ \mu_n=c(n+\gamma-1) \quad \ldots(57)$$

cannot be correct since even in the limiting case of $t\to\infty$, the variance cannot be expressed without use of hypergeometrical functions. It is therefore not possible to express variance in the non-steady case in terms of $e^{(\lambda-\mu)t}$ only. This point has been discussed in more details elsewhere (Kapur 1977).

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