ON A THEOREM OF POINCARE

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A new proof of Poincare's classical result on the presentation of a Fuchsian group with compact orbit space is given by applying Tietze transformations to a set of generators and defining relations for a general topological transformation group.

1. INTRODUCTION

Let G be a topological transformation group acting on a topological space X and let F be a locally finite closed G-covering. If (i) X is a locally path connected metric space and (ii) G is a group of isometries, then it is known (*see* Macbeath 1964) that $\langle E, \triangle \rangle$ is a presentation of G where

$$E = \{g \in G : F \cap gF \neq \phi\}$$

$$\triangle = \{[g \circ g', gg'] : F \cap gF \cap gg'F \neq \phi\},$$

$[g \circ g', gg']$ being an E-relation.

A Fuchsian group G with a compact orbit space can be viewed as a topological transformation group acting on the upper half plane D. D is a locally path connected metric space, the metric being the non-Euclidean distance between two points in D. Moreover G is a group of isometries, for it preserves non-Euclidean distance. Also G admits a Dirichlet's region K which is a closed locally finite G-covering. Therefore in view of the above result, we get the following :

$\langle E, \triangle \rangle$ is a presentation of G where

$$E = \{g \in G : K \cap gK \neq \phi\}$$

$$\triangle = \{[g \circ g', gg'] : K \cap gK \cap gg'K \neq \phi\}.$$

It follows that all relations in G are consequences of relations of type $hk = l$, where $h, k, l \in E$ and $K \cap hK \cap lK \neq \phi$. Now the last result can be restated as follows :

Theorem 1 — Let G be a Fuchsian group with compact orbit space and K a Dirichlet's region for G. Then $\langle E, \triangle \rangle$ is a presentation of G where

$$E = \{g \in G : K \cap gK \neq \phi\}$$

$$\triangle = \{hk = l : h, k, l \in E, K \cap hK \cap lK \neq \phi\}.$$

Our objective is to reduce the above big system of generators and relations to a smaller one. More precisely, our aim is to produce, with the help of this theorem and the well known Tietze transformations, a new proof of the following result of Poincare (*see* Lehner 1963) :

Theorem 2 — Let G be a Fuchsian group with compact orbit space acting on the upper half plane D and K a Dirichlet's region for G. E' denotes a set of elements of G which map K onto a neighbour and $\triangle'$ denotes a set of canonical relations which includes one for each cycle of vertices. Then E' generates G and $\triangle'$ is a complete set of defining relations for G.

2. Preliminaries

Let K be a Dirichlet's region for a Fuchsian group G with compact orbit space acting on the upper half plane. We denote by f_a the group element which maps K onto its neighbour across the edge a of K. Let $(v_1, v_2, \ldots, v_n)$ by a cycle of vertices $v_1, v_2, \ldots, v_n$ of K so that $v_1 = (a'_n, a_1)$, $v_2 = (a'_1, a_2)$, $\ldots$, $v_n = (a'_{n-1}, a_n)$ where $f_{a'} = f_{a^{-1}}$ and (a, b) is the vertex lying on the intersection of the edges labelled a and b. Corresponding to any vertex v_s of this cycle we get the canonical relation (*see* Macbeath 1961)

$$(f_{a_s} f_{a_{s+1}} \ldots f_{a_n} f_{a_1} \ldots f_{a_{s-1}})^p = 1,$$

p being the period of the cycle. The faces assembling around v_s are images of K under the group elements

$$
\begin{array}{l}
f_{a_s}, \\
f_{a_s} f_{a_{s+1}}, \\
\cdot \\
\cdot \\
\cdot \\
f_{a_s} f_{a_{s+1}} \ldots f_{a_n} f_{a_1} \ldots f_{a_{s-1}}, \\
f_{a_s} f_{a_{s+1}} \ldots f_{a_n} f_{a_1} \ldots f_{a_{s-1}} f_{a_s}, \\
f_{a_s} f_{a_{s+1}} \ldots f_{a_n} f_{a_1} \ldots f_{a_{s-1}} f_{a_s} f_{a_{s+1}}, \\
\cdot \qquad \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad \cdot \\
(f_{a_s} f_{a_{s+1}} \ldots f_{a_n} f_{a_1} \ldots f_{a_{s-1}})^{k-1} f_{a_s} f_{a_{s+1}} \ldots f_{a_n} f_{a_1} \ldots f_{a_{s-2}}.
\end{array}
$$

Since all the angles at the vertices of the cycle are positive, a finite number of faces meet at each vertex. If m is the number of faces meeting at any vertex of the cycle, then $np = m$.

Let G be a group given by the presentation

$$< a, b, c, \ldots ; P_1 = Q_1, P_2 = Q_2, \ldots >.$$

Let us call this presentation (A). Any other presentation of G can be obtained by a repeated application to (A) of the following transformations t_1, t_2, t_3, t_4 known as the Tietze transformations (*see* Magnus *et al.* 1965) :

(t_1) If the relations $S_1 = T_1$, $S_2 = T_2, \ldots$ are derivable from $P_1 = Q_1$, $P_2 = Q_2, \ldots$, then adjoin $S_1 = T_1$, $S_2 = T_2, \ldots$ to the defining relations in (A).

(t_2) If some of the relations, say $S_1 = T_1$, $S_2 = T_2, \ldots$ listed among the defining relations $P_1 = Q_1$, $P_2 = Q_2, \ldots$ are derivable from the others, then delete $S_1 = T_1$, $S_2 = T_2, \ldots$ from the defining relations in (A).

(t_3) If $M, N, \ldots$ are any words in $a, b, c, \ldots$, then adjoin the symbols $x, y, \ldots$ to the generators in (A) and adjoin the relations $x = M, y = N, \ldots$ to the defining relations in (A).

(t_4) If some of the defining relations in (A) take the form $p = V, q = W, \ldots$ where $p, q, \ldots$ are generators in (A) and $V, W, \ldots$ are words in the generators other than $p, q, \ldots$, then delete $p = V, q = W, \ldots$ from the defining relations and replace $p, q, \ldots$ by $V, W, \ldots$ respectively in the remaining relations in (A).

3. Proof of Theorem 2

Consider a cycle of period p and n vertices $v_1 = (a_n', a_1)$, $v_2 = (a_1', a_2), \ldots$, $v_n = (a_{n-1}', a_n)$. The group elements which map K onto the faces assembling around the vertex v_k are $\prod_{i=0}^{j} f_{a_{k+i}}$ where $j = 0, 1, \ldots, s$ $(s = np - 2)$ and $a_{rn+l} = a_l$, r and l being positive integers. Thus for every vertex v_k $(k = 1, \ldots, n)$ we get $np - 1$ group elements, and the group elements corresponding to all the vertices of the cycle are $A_{k,j} = \prod_{i=0}^{j} f_{a_{k+i}}$, $k = 1, \ldots, n$, $j = 0, 1, \ldots, s$. Every element $A_{k,j}$, $j \neq 0$, can be expressed as a product of two elements of the set $\{A_{k,j}\}$ and this gives rise, in view of Theorem 1, to a set of relations among the elements of $\{A_{k,j}\}$. An element $A_{k,j}$, $j \neq 0$, can be written

$$A_{k,j} = f_{a_k} f_{a_{k+1}} \ldots f_{a_{k+q}} \ldots f_{a_{k+j}} = \prod_{i=0}^{q} f_{a_{k+i}} \prod_{i=q+1}^{j} f_{a_{k+i}}$$

$$= \prod_{i=0}^{q} f_{a_{k+i}} \prod_{i=0}^{j-q-1} f_{a_{(k+q+1)+i}} = A_{k,q}\, A_{k+q+1,\, j-q-1}$$

where $q = 0, 1, \ldots, j-1$, $j = 1, \ldots, s$ and $k = 1, \ldots, n$, and also

$$A_{rn+l,j} = A_{l,j} \text{ for } A_{rn+l,j} = \prod_{i=0} f_{a_{rn+l+i}} = \prod_{i=0} f_{a_{l+i}} = A_{l,j}$$

since $a_{rn+l+i} = a_{l+i}$. Moreover the canonical relation at the vertex v_k is given by

$$\prod_{i=0}^{s+1} f_{a_{k+i}} = 1, \text{ i.e. } \prod_{i=0}^{t} f_{a_{k+i}} \prod_{i=t+1}^{s+1} f_{a_{k+i}} = 1,$$

i.e.
$$\prod_{i=0}^{t} f_{a_{k+i}} \prod_{i=0}^{s-t} f_{a_{(k+t+1)+i}} = 1,$$

i.e. $A_{k,t}A_{k+t+1,s-t} = 1$ where $t = 0, 1, \ldots, s$. Thus putting $t = 0, 1, \ldots, s$ we get $s+1$ relations of this form at the vertex v_k. Therefore corresponding to the cycle $(v_1, \ldots, v_n)$ we get the following group elements and relations :

Group elements : $A_{k,j}$, $k = 1, \ldots, n$, $j = 0, 1, \ldots, s$

Relations : (i) $A_{k,j} = A_{k,q}A_{k+q+1,j-q-1}$, $j = 1, \ldots, s$, $q = 0, 1, \ldots, j-1$, $k = 1, \ldots, n$

(ii) $A_{k,t}A_{k+t+1,s-t} = 1$, $t = 0, 1, \ldots, s$, $k = 1, \ldots, n$

where $A_{rn+l,j} = A_{l,j}$. In future the ranges of values of k, j, q and t will be understood to be as above.

Since $K \cap xK$ where $x \in \{1, A_{k,j}\}$ has at least a vertex of K, $K \cap xK \neq \phi$. Thus $\{1, A_{k,j}\} \subset E = \{g \in G : K \cap gK \neq \phi\}$. Also all the relations listed above belong to $\triangle = \{hk = l : h, k, l \in E, K \cap hK \cap lK \neq \phi\}$, because

$$A_{k,j} = A_{k,q}A_{k+q+1,j-q-1} \in \triangle$$

since $A_{k,j}, A_{k,q}, A_{k+q+1,j-q-1} \in E$ and $K \cap A_{k,j}K \cap A_{k,q}K \neq \phi$, and similarly $A_{k,t}A_{k+t+1,s-t} = 1 \in \triangle$ because $1, A_{k,t}, A_{k+t+1,s-t} \in E$ and $K \cap A_{k,t}K \cap lK \neq \phi$.

Thus we have explicitly written down the generators and relations corresponding to the cycle $(v_1, \ldots, v_n)$ which constitute a subset of $\langle E, \triangle \rangle$. The union of the sets of generators and relations obtained by considering all the cycles of K is $\langle E, \triangle \rangle$.

We now apply the Tietze transformations to the above subset of $\langle E, \triangle \rangle$ with a certain restriction. The transformations (t_1), (t_2), (t_3) are clearly applicable to any subset of $\langle E, \triangle \rangle$, for these do not involve elimination of any generator and subsequent substitution of the 'value' of the eliminated generator in other relations in which it may occur. But we cannot, and so we do not, apply (t_4) to eliminate the generators $A_{k,0}$, $k = 1, \ldots, n$, because these may occur in relations other than those listed above. The other generators, i.e. $A_{k,j}$, $j \neq 0$, of the above subset of $\langle E, \triangle \rangle$ never occur elsewhere, and so we can try (t_4) to eliminate them.

By (t_2) we eliminate $A_{k,j} = A_{k,q}A_{k+q+1,j-q-1}$ $j = 2, \ldots, s$, $q = 0, 1, \ldots, j-2$, $k = 1, \ldots, n$ as these are derivable from $A_{k,j} = A_{k,j-1}A_{k+j,0}$ ($q = j-1$). For, let $A_{k,j} = A_{k,q}A_{k+q+1,j-q-1}$ be a typical relation such that $q < j-1$. This relation is derivable from $A_{k,j} = A_{k,j-1}A_{k+j,0}$, $A_{k,j-1} = A_{k,j-2}A_{k+j-1,0}$, $\ldots$, $A_{k,q+1} = A_{k,q}A_{k+q+1,0}$ and $A_{k+q+1,j-q-1} = A_{k+q+1,j-q-2}A_{k+j,0}$, $A_{k+q+1,j-q-2} = A_{k+q+1,j-q-3}A_{k+j-1,0}$, $\ldots$, $A_{k+q+1,1} = A_{k+q+1,0}A_{k+q+2,0}$ all of which are of type $A_{k,j} = A_{k,j-1}A_{k+j,0}$. Thus we are left with the following relations :

$A_{k,j} = A_{k,j-1}A_{k+j,0}$, $j = 1, \ldots, s$, $k = 1, \ldots, n$ and $A_{k,t}A_{k+t+1,s-t} = 1$, $t = 0, 1, \ldots, s$, $k = 1, \ldots, n$.

Next we apply (t_1) to adjoin $A_{k,e-1}A_{k+e,0}A_{k+e+1,s-e} = 1$, $e = 1, \ldots, s$. For, a typical relation of this form is derivable from $A_{k,e} = A_{k,e-1}A_{k+e,0}$ and $A_{k,e}A_{k+e+1,s-e} = 1$ which belong to the set of relations left after the application of t_2. Now we apply t_2 again to delete $A_{k,t}A_{k+t+1,s-t} = 1$, $t = 1, \ldots, s$, $k = 1, \ldots, n$, since a typical relation of this form is derivable from $A_{k,t} = A_{k,t-1}A_{k+t,0}$ and $A_{k,t-1}A_{k+t+1,s-t} = 1$ which belong to the set of relations of the other two types. Thus at this stage we have the following relations :

$A_{k,j} = A_{k,j-1}A_{k+j,0}$, $j = 1, \ldots, s$, $k = 1, \ldots, n$, $A_{k,0}A_{k+1,s} = 1$, $k = 1, \ldots, n$ and $A_{k,e-1}A_{k+e,0}A_{k+e+1,s-e} = 1$, $e = 1, \ldots, s$, $k = 1, \ldots, n$.

We once again apply (t_2) to delete $A_{k,e-1}A_{k+e,0}A_{k+e+1,s-e} = 1$, $k = 1, \ldots, n$, $e = 1, \ldots, s$, for any typical relation $A_{k,e-1}A_{k+e,0}A_{k+e+1,s-e} = 1$ is derivable from

$$A_{k,e-1} = A_{k,e-2}A_{k+e-1,0},\ A_{k,e-2} = A_{k,e-3}A_{k+e-2,0},\ \ldots,\ A_{k,1} = A_{k,0}A_{k+1,0},$$

$$A_{k+e+1,s-e} = A_{k+e+1,s-e-1}A_{k+s+1,0},\ A_{k+e+1,s-e-1} = A_{k+e+1,s-e-2}A_{k+s,0},\ \ldots,$$

$A_{k+e+1,1} = A_{k+e+1,0}A_{k+e+2,0}$ and $A_{k,0}A_{k+1,s} = 1$ together with $A_{k+1,s} = A_{k+1,s-1}A_{k+s+1,0}$, $A_{k+1,s-1} = A_{k+1,s-2}A_{k+s,0}$, $\ldots$, $A_{k+1,1} = A_{k+1,0}A_{k+2,0}$, and all these are of types $A_{k,j} = A_{k,j-1}A_{k+j,0}$, $j = 1, \ldots, s$, $k = 1, \ldots, n$ and $A_{k,0}A_{k+1,s} = 1$, $k = 1, \ldots, n$, which are the only relations we are now left with.

Now for a fixed value of k, say k_1, we get the relations $A_{k_1+1,1} = A_{k_1+1,0}A_{k_1+2,0}$ $A_{k_1+1,2} = A_{k_1+1,1}A_{k_1+3,0}$, $\ldots$, $A_{k_1+1,s} = A_{k_1+1,s-1}A_{k_1+s+1,0}$ and $A_{k_1,0}A_{k_1+1,s} = 1$. Using (t_4) we then eliminate the generators $A_{k_1+1,1}$, $A_{k_1+1,2}$, $\ldots$, $A_{k_1+1,s}$ and the corresponding relations one by one, and we get the single relation $A_{k_1,0}A_{k_1+1,0}, \ldots, A_{k_1+s+1,0} = 1$. Since the above result is valid for every value of k, we ultimately get the following generators and relations :

Generators : $A_{k,0}$, $k = 1, \ldots, n$

$$\text{Relations}: \prod_{i=0}^{s+1} A_{k+i,0} = 1, k = 1, \ldots, n \qquad \ldots(X)$$

Any relation of (X) is derivable from any other. For example

$$A_{k_1,0}A_{k_1+1,0} \ldots A_{k_1+s+1,0} = 1$$

is derivable from $A_{k_1+1,0}A_{k_1+2,0} \ldots A_{k_1+s+2,0} = 1$ by pre-multiplication by $A_{k_1,0}$ and post-multiplication by $A^{-1}_{k_1,0}$ and remembering that $A_{k_1+s+2,0} = A_{pn+k_1,0} = A_{k_1,0}$. Hence by using ($t_2$) we can delete all but any one of the relations (X). So finally we have the following generators and relation only :

$$\text{Generators}: A_{k,0}, k = 1, \ldots, n, \text{ i.e. } f_{a_1}, \ldots, f_{a_n} \qquad \ldots(Y)$$

$$\text{Relation}: \prod_{i=0}^{s+1} A_{1+i,0} = 1, \text{ i.e. } f_{a_1} \ldots f_{a_{s+2}} = 1 \qquad \ldots(Z)$$

Each of the generators of (Y) maps K onto a neighbour and the relation (Z) is the canonical relation at v_1. Thus the set of generators and relations that arise from the cycle $(v_1, \ldots, v_n)$ reduces to one consisting of generators which map K onto neighbours and a relation which is the canonical relation at one of the vertices of the cycle. For an inessential cycle $p = 1$, and since $np \geqslant 3$ (at least three faces meet at a vertex of K), the arguments given above do not collapse so long we have $np \geqslant 3$. Further, the cycles of K are disjoint and their union is the set of all the vertices of K and each cycle belongs to either of the two categories — essential ($p \geqslant 2$) and inessential ($p = 1$). Moreover, the totality of the generators and relations that arise from all the cycles of K is $\langle E, \triangle \rangle$, a presentation of G. Therefore considering every cycle separately and applying the above process we see the validity of the statement of Theorem 2.

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