ON STRONG MATRIX SUMMABILITY

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In this note, we establish necessary and sufficient conditions for strong matrix summability of series Σu_n. The paper generalizes a theorem of Hyslop (1952) on strong Cesàro summability of infinite series and theorems of Prasad (1966) and Prasad and Mittal (1978) on strong Nörlund summability.

§1. Let Σu_n be a given infinite series with sequence of partial sums $\{s_n\}$. Let $\| T \| \equiv (a_{n,k})$ be an infinite rectangular matrix with real constants. Then sequence-to-sequence transformation

$$t_n = \sum_{k=0}^{\infty} a_{n,k}\, s_k, \quad n = 0, 1, 2, \ldots, \qquad \ldots(1.1)$$

defines the T-transform of the sequence $\{s_n\}$. The series Σu_n is said to be T-summable to s, if $\lim_{n\to\infty} t_n = s$.

If matrix elements $a_{n,k} = 0$, for every $k > n$, then the matrix is called triangular matrix. The matrix T — reduces to Nörlund (1919) matrix generated by the sequence of coefficients $\{p_n\}$, if

$$a_{n,k} = \begin{cases} \dfrac{p_{n-k}}{P_n}, & \text{if } k \leqslant n\ ; \\ 0, & \text{if } k > n\ ; \end{cases}$$

where

$$P_n = \sum_{k=0}^{n} p_k \neq 0.$$

§2. Srivastava (1960) has given the following definition for strong matrix summability :

Definition 1 — The series Σu_n is said to be strongly summable with index q, by the T — process or summable $[T, q]$, $q \geqslant 1$, to the sums s, if

$$\sum_{r=0}^{n} r^q \mid t_r - t_{r-1} \mid^q = o(n), \text{ as } n \to \infty\ ; \qquad \ldots(2.1)$$

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and

$$t_n \to s, \text{ as } n \to \infty \; ; \qquad \ldots(2.2)$$

where t_n is defined as in (1.1).

If the matrix T is a Nörlund matrix, then this definition reduces to the definition of strong Nörlund summability $[N, p_n, q]$ of the series $\Sigma\, u_n$.

The authors Prasad and Mittal (1978) have proved a following necessary and sufficient condition for $[N, p_n, q]$ summability :

Theorem A — If $p_n > 0$, and satisfies

$$(n+1)\, p_n = O\,(P_n) \qquad \ldots(2.3)$$

then the necessary and sufficient conditions for $\Sigma\, u_n$ to be summable $[N, p_n, q]$, $q \geqslant 1$, to the sum s, is that

$$\sum_{r=0}^{n} \left\{ \frac{(r+1)\, p_r}{P_r} \right\}^q \mid T_r - s \mid^q = o(n) \text{ as } n \to \infty \; ; \qquad \ldots(2.4)$$

and

$$t_n \to s, \text{ as } n \to \infty \; ; \qquad \ldots(2.5)$$

where

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k}\, s_k, \qquad \ldots(2.6)$$

and

$$\left. \begin{aligned} T_n &= \frac{1}{p_n} \sum_{k=0}^{n} s_k \nabla p_{n-k} \qquad (\nabla p_{n-k} \equiv p_{n-k} - p_{n-k-1}) \\ &= \frac{1}{p_n} \sum_{k=0}^{n} u_k p_{n-k}. \end{aligned} \right\} \qquad \ldots(2.7)$$

In this note, we extended Theorem A to strong matrix summability.
We prove :

Theorem 1 — Let $\| T \| \equiv (a_{n,k})$ be an infinite triangular matrix and $A_{n,k} = \sum\limits_{r=k}^{n} a_{n,r}$ with $a_{n,k} \geqslant 0$, $A_{n,0} = 1$ and let $a_{n,k}$ also satisfy

$$a_{n-1,k} \geqslant (1 + a_{n,0})\; a_{n,k+1}, \text{ for } 0 \leqslant k \leqslant n-1,\; n \geqslant 1\;; \qquad ...(2.8)$$

$$a_{n,0} = O(1/n) \qquad ...(2.9)$$

then the necessary and sufficient conditions for $\Sigma\, u_n$ with $s_n = o(n)$, to be summable $[T, q]$ to the sum s, are that

$$\sum_{r=0}^{n} [r a_{r,0}\,(1 + a_{r,0})]^q \mid T_r - s \mid^q = o(n), \text{ as } n \to \infty\;; \qquad ...(2.10)$$

and

$$t_n \to s, \text{ as } n \to \infty\;; \qquad ...(2.11)$$

where t_n is defined as in (1.1) and

$$\left.\begin{aligned} T_n &= \frac{1}{a_{n,0}} \sum_{k=0}^{n} s_k\, \triangle_k\, a_{n,k} \\ &\qquad\qquad (\triangle_k\, a_{n,k} \equiv a_{n,k} - a_{n,k+1}) \\ &= \frac{1}{a_{n,0}} \sum_{k=0}^{n} u_k\, a_{n,k}. \end{aligned}\right\} \qquad ...(2.12)$$

In proving Theorem 1, we require the following lemma :

Lemma 1 (Varshney 1977) — If $a_{n,k} \geqslant 0$, satisfies condition (2.8) and (2.9) and if

$$B_{r,k} = A_{r-1,k} - (1 + a_{r,0})\, A_{r,k+1}, \text{ for } 0 \leqslant k \leqslant r-1\;;\; r \geqslant 1\;;$$

then

$$B_{r,k} \geqslant 0\;;\; B_{r,k} \geqslant B_{r,k+1} \text{ and } B_{r,k} \leqslant \frac{C}{(r+1)^2}.$$

Proof of Theorem 1 : Without any loss of generality, we may assume that the sum of the series is zero. First by using (1.1) and (2.12), we have

$$T_r = \frac{1}{a_{r,0}} \sum_{k=0}^{r} s_k\, \triangle_k\, a_{r,k}$$

$$= \frac{1}{a_{r,0}} \left(\sum_{k=0}^{r} s_k\, a_{r,k} - \sum_{k=0}^{r} s_k a_{r,k+1} \right)$$

$$= \frac{1}{a_{r,0}} \sum_{k=0}^{r} s_k(a_{r,k} - a_{r-1,k})$$

$$+ \frac{1}{a_{r,0}} \sum_{k=0}^{r-1} s_k \left[a_{r-1,k} - (1 + a_{r,0})\, a_{r,k+1}\right]$$

$$+ \sum_{k=0}^{r-1} s_k a_{r,k+1}$$

$$= \frac{1}{a_{r,0}} \left[(t_r - t_{r-1}) + \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) \right]$$

$$+ \sum_{k=0}^{r-1} s_{k+1}\, a_{r,k+1} - \sum_{k=0}^{r-1} u_{k+1}\, a_{r,k+1}$$

$$= \frac{1}{a_{r,0}} \left[(t_r - t_{r-1}) + \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) \right]$$

$$+ \sum_{k=0}^{r} a_{r,k} s_k - \sum_{k=1}^{r} a_{r,k} u_k$$

$$= \frac{1}{a_{r,0}} \left[(t_r - t_{r-1}) + \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) \right] + t_r - a_{r,0} T_r$$

Thus

$$(1 + a_{r,0})\, T_r = \frac{1}{a_{r,0}} \left[(t_r - t_{r-1}) + \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) + a_{r,0} t_r \right]$$

or

$$r a_{r,0}(1 + a_{r,0})\, T_r = r(t_r - t_{r-1}) + r \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) + r a_{r,0}\, t_r \qquad \ldots(2.13)$$

or

$$\sum_{r=0}^{n} \{r a_{r,0}(1 + a_{r,0})\, |\, T_r \,|\}^q = \sum_{r=0}^{n} \{\, |\, r(t_r - t_{r-1})$$

$$+ r \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) + r a_{r,0}\, t_r \,|\, \}^q$$

Now, using Minkowski's inequality, (2.1), (2.2) and (2.9), we get

$$[\sum_{r=0}^{n} \{ra_{r,0}(1 + a_{r,0}) \mid T_r \mid\}^q]^{1/q}$$

$$= [\sum_{r=0}^{n} \mid r(t_r - t_{r-1}) + r \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) + ra_{r,0}\, t_r \mid^q]^{1/q}$$

$$\leqslant [\sum_{r=0}^{n} r^q \mid t_r - t_{r-1} \mid^q]^{1/q} + [\sum_{r=0}^{n} r^q \mid \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) \mid^q]^{1/q}$$

$$+ [\sum_{r=0}^{n} (ra_{r,0})^q \mid t_r \mid^q]^{1/q}$$

$$= o(n^{1/q}) + [\sum_{r=0}^{n} r^q \{\sum_{k=0}^{r-1} o(k) \mid B_{r,k} - B_{r,k+1} \mid\}^q]^{1/q}$$

$$+ [\sum_{r=0}^{n} (ra_{r,0})^q \mid t_r \mid^q]^{1/q}$$

$$\leqslant o(n^{1/q}) + [\sum_{r=0}^{n} o(r^{2q}) \{\sum_{k=0}^{r-1} (B_{r,k} - B_{r,k+1})\}^q]^{1/q}$$

$$= o(n^{1/q}) + [\sum_{r=0}^{n} o(r^{2q})\, B^q_{r,0}]^{1/q}$$

$$\leqslant o(n^{1/q}) + (\sum_{r=0}^{n} o(1))^{1/q}$$

$$= o(n^{1/q}) \qquad \ldots(2.14)$$

in view of lemma 1 and $s_k = o(k)$. Hence the conditions (2.10) and (2.11) are necessary. We now prove that conditions (2.10) and (2.11) are sufficient.

Again, by using Minkowski's inequality, (2.9), (2.10), (2.11) and (2.13), we have

$$\mid r(t_r - t_{r-1}) \mid^q = \mid ra_{r,0}(1 + a_{r,0})\, T_r - r \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) - ra_{r,0}\, t_r \mid^q$$

or

$$[\sum_{r=0}^{n} \mid r(t_r - t_{r-1}) \mid^q]^{1/q} = [\sum_{r=0}^{n} \mid ra_{r,0}(1 + a_{r,0})\, T_r - r \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) - ra_{r,0}t_r \mid^q]^{1/q}$$

$$\leqslant [\sum_{r=0}^{n} \{ra_{r,0}(1 + a_{r,0})\}^q \mid T_r \mid^q]^{1/q}$$

$$+ [\sum_{r=0}^{n} r^q \mid \sum_{k=0}^{r-1} s_k(B_{r,k} - B_{r,k+1}) \mid^q]^{1/q}$$

$$+ [\sum_{r=0}^{n} (ra_{r,0})^q \mid t_r \mid^q]^{1/q}$$

$$= o(n^{1/q}) + [\sum_{r=0}^{n} r^q \{\sum_{k=0}^{r-1} o(k) \mid B_{r,k} - B_{r,k+1} \mid \}^q]^{1/q}$$

$$\leqslant o(n^{1/q}) + [\sum_{r=0}^{n} o(r^{2q}) \{\sum_{k=0}^{r-1} (B_{r,k} - B_{r,k+1})\}^q]^{1/q}$$

$$\leqslant o(n^{1/q}) + o(n^{1/q})$$

$$= o(n^{1/q}) \qquad \ldots(2.15)$$

in view of lemma 1 and $s_k = o(k)$.

Thus, combining (2.14) and (2.15), this completes the proof of Theorem 1.

But, we know

$$\mid T_r \mid < (1 + a_{r,0}) \mid T_r \mid < 2 \mid T_r \mid \qquad \ldots(2.16)$$

in view of (2.9).

Therefore, we have

$$\sum_{r=0}^{n} (ra_{r,0})^q \mid T_r - s \mid^q = o(n) \Leftrightarrow$$

$$\sum_{r=0}^{n} \{ra_{r,0}(1 + a_{r,0})\}^q \mid T_r - s \mid^q = o(n). \qquad \ldots(2.17)$$

In view of Theorem 1, and (2.17), we state the following Theorem :

Theorem 2 — Let $\| T \| \equiv (a_{n,k})$ be an infinite triangular matrix and $A_{n,k} = \sum_{r=k}^{n} a_{n,r}$ with $a_{n,k} \geqslant 0$, $A_{n,0} = 1$ and let $a_{n,k}$ also satisfy (2.8) and (2.9), then the necessary and sufficient conditions for Σu_n with $s_n = o(n)$, to be summable $[T, q]$, $q \geqslant 1$, to the sum s, are that

$$\sum_{r=0}^{n} (ra_{r,0})^q \mid T_r - s \mid^q = o(n), \text{ as } n \to \infty, \text{ and condition (2.11).} \qquad \ldots(2.18)$$

Theorem 2 is an extension of Theorem A and also it includes some other results by Hyslop (1951) and Prasad (1966).

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