# HYPERSURFACES IN AN ALMOST PARACONTACT MANIFOLD

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We discussed several aspects of an almost paracontact manifold. Hypersurfaces of the almost paracontact manifold have been studied under different conditions both affinely and metrically.

§1. An almost paracontact structure on a differentiable manifold $M_n$ is defined by (Sato 1976)

$$F^2 = I - A \otimes T \qquad \text{...(1.1)}$$

$$\text{rank } ((F)) = n - 1,\ FT = 0,\ AF = 0,\ A(T) = 1. \qquad \text{...(1.2)}$$

where $F$ is a tensor field of type (1, 1), $T$ a vector field and $A$ a 1-form. The manifold equipped with the almost paracontact structure is called almost paracontact manifold.

Let us define a Riemannian metric $G$ on the almost paracontact manifold by

$$G(F\tilde{X}, F\tilde{X}) = -\ G(\tilde{X}, \tilde{Y}) + A(\tilde{X})\, A(\tilde{Y}) \qquad \text{...(1.3a)}$$

$$G(T, \tilde{X}) = A(\tilde{X}) \qquad \text{...(1.3b)}$$

where $\tilde{X}$ and $\tilde{Y}$ are arbitrary vector fields on $M_n$. An almost paracontact manifold with the above metric is called an almost hyperbolic contact manifold (Sinha and Kalpana 1978). The tensor fields $'F$ of type (0, 2) defined by

$$'F(\tilde{X}, \tilde{Y}) = G(F\tilde{X}, \tilde{Y})$$

is skew-symmetric in $\tilde{X}$, $\tilde{Y}$.

§2. Let $M_{n-1}$ be the hypersurface imbedded in $M_n$ by the imbedding $b : M_{n-1} \rightarrow M_n$ and $B$ the Jacobian of $b$, that is $B : T_p(M_{n-1}) \rightarrow T_{bp}(M_n)$ for any point $p$ of $M_{n-1}$. Consider a vector field $N$ on $M_n$ which does not belong to $TM_{n-1}$ so that $N$ is nowhere tangent to $M_{n-1}$. Since $B$ is one-one, it is possible to obtain an inverse map $B^{-1} : T_{bp}(M_n) \rightarrow T_p(M_{n-1})$ for any point $p$ of $M_{n-1}$ and a 1-form $N^*$ on $M_n$ such that

$$\left.\begin{array}{l} B^{-1} B = I,\ BB^{-1} = I - N^* \otimes N \\ N^* B = 0,\ B^{-1} N = 0,\ N^* (N) = 1. \end{array}\right\} \qquad \ldots(2.1)$$

If the enveloping manifold $M_n$ be a hyperbolic contact manifold with metric $G$. the metric $g$ induced on the hypersurface $M_{n-1}$ will be such that

$$G(BX, BY) = g(X, Y)$$

where $X$ and $Y$ are arbitrary vector fields on $M_{n-1}$. In what follows, we will restrict ourselves to only two cases viz. either $T$ is nowhere tangent to $M_{n-1}$ or $T$ is everywhere tangent to $M_{n-1}$.

*Theorem* 2.1 — Let $M_{n-1}$ be a hypersurface in an almost paracontact manifold $M_n$. Then a tensor field $f$ of type (1,1) can be defined on $M_{n-1}$ such that

$$f^3 - f = 0$$

called a para $f$-structure (Singh and Vohra 1972) and the rank $((f)) = n - 1$ or $n - 3$.

PROOF : Let us put

$$f \stackrel{def}{=} B^{-1}FB.$$

Then using (2.1) we have

$$\begin{aligned} f^3X &= B^{-1} FBB^{-1} FBB^{-1} FBX \\ &= B^{-1} F(I - N^* \otimes N) F(I - N^* \otimes N) FBX \\ &= B^{-1} F(I - N^* \otimes N) (BX - A(BX)\, T - N^* (FBX)\, FN) \\ &= B^{-1} FBX - N^* (FBX)\, B^{-1} F^2 N + A(BX)\, N^* TB^{-1} FN \\ &\qquad + N^* (FBX)\, N^* (FN)\, B^{-1} FN. \end{aligned}$$

The term $N^* (FN)$ may be taken as $G(FN, N)$ which vanishes because $'F$ is skew-symmetric and thus we have

$$f^3X = f X + N^* (FBX)\, A(N)\, B^{-1}\, T + A(BX)\, N^*\, TB^{-1} FN.$$

Now let $T$ be nowhere tangent to $M_{m-1}$ so that $T$ plays the role of $N$ and hence $f^3 - f = 0$. In this case $fX = 0$ implies $BX = A(BX)\, N$ which yields $X = 0$. Hence the rank of $f$ is $n - 1$.

If, on the other hand, $T$ is everywhere tangent to $M_{n-1}$ so that $N^*\, T = A(N) = 0$ which yields $f^3 - f = 0$. Since $T$ is tangential to $M_{n-1}$ there exists a non-null vector field $X_1$ on $M_{n-1}$ such that $BX_1 = T$. So $fX_1 = 0$. Since $FN$ is orthogonal to $N$, it is tangential to $M_{n-1}$ and therefore there exists a non-null vector field $X_2$ on $M_{n-1}$ such that $FN = BX_2$ which implies $fX_2 = 0$. Obviously $X_1$ and $X_2$

are orthogonal. Let $X_3$ be a non-null vector field on $M_{n-1}$, which is tangential to $M_{n-1}$ and orthogonal to $X_1$ and $X_2$. Then there exists a vector field $X_4$ such that $F(BX_4) = BX_3$ which implies $fX_4 = X_3$. Let $fX_3 = 0$. Then $f^2X_4 = 0$ which gives $X_3 = 0$. Hence there are only two non-null orthogonal vector fields $X_1$ and $X_2$ on $M_{n-1}$ such that $fX_1 = fX_2 = 0$. Thus rank $((f)) = n - 3$.

*Corollary* 2.1 — In order that the tensor field $B^{-1}FB$ on the hypersurface $M_{n-1}$ of $M_n$ with the almost paracontact structure $(F, T, A)$ may define an almost product structure on $M_{n-1}$, it is necessary and sufficient that $T$ is nowhere tangent to $M_{n-1}$.

PROOF : The sufficient part of the proof is provided in the proof of the previous theorem. To prove the necessary part we suppose that $T$ is everywhere tangent to $M_{n-1}$. Then, from the proof of the previous theorem, we have $B^{-1}FB$ as a para $f$-structure of rank $n - 3$. Consequently $B^{-1}FB$ is not an almost product structure on $M_{n-1}$.

§3. *Lemma* 3.1 — Let $T'$ be a non-zero vector field on $M_n$, which is nowhere in the $T$ direction. Then we can define a non-singular tensor field $\mu$ of type (1, 1) on $M_n$ such that

$$\mu T' = T. \qquad \ldots(3.1)$$

PROOF : It is easy to see that $T', FT', T$ are linearly independent. Let us have $n - 3$ linearly independent vector field $\overset{a}{P}(a = 1, 2, \ldots, n-3)$ on $M_n$ such that $\{\overset{a}{P}, T', FT', T\}$ is a basis of the Lie-algebra of the vector fields on $M_n$. The dual basis is $\{\underset{a}{A}, A', A'', A'''\}$, say. We define a tensor field $\mu$ of type (1, 1) by (Blair and Ludden 1969)

$$\mu(\tilde{X}) = \underset{a}{A}(\tilde{X})\,\overset{a}{P} + A'(\tilde{X})\,T + A'''(\tilde{X})\,T' + A''(\tilde{X})\,FT'. \qquad \ldots(3.2)$$

Putting $\tilde{X} = T'$ in it we obtain $\mu T' = T$.

*Theorem* 3.1 — The almost paracontact structure on $M_n$ is not unique.

PROOF : Let $(F, T, A)$ be an almost paracontact structure on $M_n$. Let $T'$ be a non-zero vector field nowhere in the $T$ direction. Then by Lemma 3.1, we have a tensor field $\mu$ of type (1, 1) such that

$$\mu T' = T.$$

If we define a tensor field $F'$ and a 1-form $A'$ by

$$\mu F'\tilde{X} = F\mu\tilde{X}$$

$$A'(\tilde{X}) = A(\mu\tilde{X})$$

then we have

$$\mu F'^2 \tilde{X} = F \mu F' \tilde{X} = F^2 \mu \tilde{X} = \mu \tilde{X} - A(\mu \tilde{X})\, T = \mu \tilde{X} - \mu A'(\tilde{X})\, T$$

yielding

$$F'^2 \tilde{X} = \tilde{X} - A'(\tilde{X})\, T'.$$

Thus $(F', T', A')$ is another almost paracontact structure.

*Theorem* 3.2 — Let $M_{n-1}$ be a hypersurface in an almost paracontact manifold $M_n$ and let there exist a non-zero vector-field $T'$ on $M_n$ which is nowhere tangent to $M_{n-1}$. Then $M_{n-1}$ is an almost product manifold with the structure tensor $B^{-1}F'B$ where $F'$ is given in the proof of Theorem 3.1.

PROOF : Using Theorem 3.1 and Corollary 2.1 we can prove it.

*Theorem* 3.3 — The tensor field $f$ defined by $f = \frac{1}{\sqrt{2}}(F - F')$ where $(F, T, A)$ and $(F', T', A')$ are almost paracontact structures as defined in Theorem 3.1, defines a para $f$-structure on $M_n$.

PROOF : We have

$$\begin{aligned}(f^3 - f)(T) &= (f^2 - I)(fT) \\ &= -\frac{1}{\sqrt{2}}\left(\tfrac{1}{2}(F^2 + F'^2 - FF' - F'F) - I\right)(F'T) \\ &= -\frac{1}{2\sqrt{2}}(F'FF'T) \\ &= 0\end{aligned}$$

where we have used the fact $F'T = FT'$ which can be proved by using $\mu T = T'$ and $\mu(FT') = FT'$ as derived from (3.2). Similarly we can prove that $(f^3 - f)T' = 0$. Now we have

$$f(FT') = \frac{1}{\sqrt{2}}(F^2T - F'^2T) = \frac{1}{\sqrt{2}}(T' - T)$$

where

$$\begin{aligned}f^3(FT) &= \frac{1}{2\sqrt{2}}(F^2 + F'^2 - FF' - F'F)(T' - T) \\ &= \frac{1}{\sqrt{2}}(T' - T) \\ &= f(FT').\end{aligned}$$

Lastly, for a vector field $\tilde{U}$ which is nowhere in the directions of $T$, $T'$ and $FT'$ we have from (3.2),

$$\mu\tilde{U} = \underset{a}{A}(\tilde{U}) \overset{a}{P} = \tilde{U}$$

$$\mu(F\tilde{U}) = \underset{a}{A}(\tilde{U})\, \mu\, (F\overset{a}{P}) = \underset{a}{A}(\tilde{U})\, A''\, (F\overset{a}{P}) = \underset{a}{A}(\tilde{U})\, F\overset{a}{P} = F\tilde{U}.$$

From these two equations we obtain

$$F\mu\tilde{U} = F\tilde{U}$$

whence

$$\mu^{-1}F\mu\tilde{U} = \mu^{-1}F\tilde{U} = F\tilde{U}$$

or

$$F'\tilde{U} = F\tilde{U}.$$

Consequently $f\tilde{U} = 0$. Summing up the three results we see that $f$ defines a para $f$-structure on $M_n$. Since

$$fT = -\frac{1}{\sqrt{2}}\,(F'T) = -\frac{1}{\sqrt{2}}\,(FT')$$

$$fT' = \frac{1}{\sqrt{2}}\,(FT')$$

and

$$f\tilde{U} = 0$$

where $\tilde{U}$ is nowhere in the directions of $T$, $T'$, $FT'$ we observe that the range of $f$ is spanned by the vector fields $T' - T$ and $FT'$ and hence the rank of $f$ is 2.

§4. *Theorem* 4.1 — The almost hyperbolic contact structure on $M_n$ is not unique.

PROOF: Let $(F, T, A, G)$ be an almost hyperbolic contact structure on $M_n$. Let $F'$, $T'$, $A'$ be defined as in Theorem 3.1 and let us define a metric $G'$ by

$$G'(\tilde{X}, \tilde{Y}) = G(\mu\tilde{X}, \mu\tilde{Y}).$$

Theorem will be proved if we only show that $G'$ satisfies (1.3a) and (1.3b)

$$\begin{aligned} G'(F'\tilde{X}, F'\tilde{Y}) &= G(\mu F'\tilde{X}, \mu F'\tilde{Y}) \\ &= G(F\mu\tilde{X}, F\mu\tilde{Y}) \\ &= -G(\mu\tilde{X}, \mu\tilde{Y}) + A(\mu\tilde{X})\, A(\mu\tilde{Y}) \\ &= -G'(\tilde{X}, \tilde{Y}) + A'(\tilde{X})\, A'(\tilde{Y}) \end{aligned}$$

$$
\begin{aligned}
G'(T', \tilde{X}) &= G(\mu T', \mu\tilde{X}) \\
&= G(T, \mu\tilde{X}) \\
&= A'(\tilde{X}).
\end{aligned}
$$

*Theorem* 4.2 — The hypersurface $M_{n-1}$ of an almost hyperbolic contact manifold $M_n(F, T, A, G)$ is an almost hyperbolic Hermite manifold (Dube 1978) if $T$ is orthogonal to $M_{n-1}$.

PROOF : First, in the light of corollary 2.1, we observe that $M_{n-1}$ has an almost product structure $B^{-1} FB = f$.

Now we have

$$
\begin{aligned}
g(fX, fY) &= G(BfX, BfY) \\
&= G(FBX, FBY) \\
&= -\, G(BX, BY) + A(BX)\, A(BY) \\
&= -\, G(BX, BY). \\
&= -\, g(X, Y)
\end{aligned}
$$

*Theorem* 4.3 — Let $M_{n-1}$ be a hypersurface of a hyperbolic contact manifold $M_n(F, T, A, G)$. Let $T'$ be a nonzero vector field on $M_n$ orthogonal to $T$ and to $M_{n-1}$. Then $M_{n-1}$ has a hyperbolic Hermite structure.

PROOF : This theorem is a consequence of theorems 4.1 and 4.2 together with the fact that $T'$ is orthogonal to $M_{n-1}$ under the metric $G'$, which can be proved as follows :

Let us assume

$$BX = \tilde{X} + \lambda T$$

where $G(\tilde{X}, T) = 0$ and $G(BX, T') = 0$. We can set such that $\mu\tilde{X} = \tilde{X}$ and thus

$$
\begin{aligned}
G'(BX, T') &= G(\mu BX, \mu T') \\
&= G(\mu\tilde{X} + \lambda T', T) \\
&= G(\tilde{X}, T) \\
&= 0.
\end{aligned}
$$

Now let $(F, T, A, G)$ and $(F', T', A', G')$ be a pair of hyperbolic contact structures on $M_n$ as defined earlier and let $'F$ and $'F'$ be their 2-forms. Then

$$'F(\tilde{X}, \tilde{Y}) = G'(F'\tilde{X}, \tilde{Y}) = G(F\mu\tilde{X}, \mu\tilde{Y}) = 'F(\mu\tilde{X}, \mu\tilde{Y})$$

which shows that the 2-form is invariant under $\mu$.

The almost paracontact manifold is said to be normal if

$$N_F - (dA) \otimes T = 0$$

where $N_F$ is the Nijenhuis tensor of $F$.

If the fundamental 2-form $'F$ of the hyperbolic contact manifold $M_n$ be closed, we call $M_n$ a quasi-normal hyperbolic contact manifold (Blair 1967).

*Theorem* 4.4 — The almost product structure of the almost product hypersurface $M_{n-1}$ in the normal almost paracontact manifold $M_n$ is integrable. Also, if $M_n$ is a quasi-normal hyperbolic contact manifold then $M_{n-1}$ is an almost decomposable hyperbolic Hermite manifold.

PROOF : If $N_f$ and $N_F$ denote the Nijenhuis tensors of the structures $f$ and $F$ respectively then we have

$$\begin{aligned} N_f(X, Y) &= [fX, fY] + [X, Y] - f[fX, Y] - f[X, fY] \\ &= B^{-1}[FBX, FBY] + B^{-1}[BX, BY] - B^{-1}F[FBX, BY] \\ &\qquad -B^{-1} F[BX, FBY] \\ &= B^{-1}([FBX, FBY] + F^2[BX, BY] - F[FBX, BY] \\ &\qquad - F[BX, FBY] + A[BX, BY]\, T) \\ &= B^{-1}(N_F(BX, BY) - (dA)(BX, BY)\, T) \\ &= 0. \end{aligned}$$

For the second part we have

$$\begin{aligned} (d'f)(X, Y, Z) &= X\,'f(Y, Z) - Y\,'f(X, Z) + Z\,'f(X, Y) \\ &\quad - 'f([X, Y], Z) + 'f([X, Z], Y) - 'f([Y, Z], X) \\ &= BX\,'F(BY, BZ) - BY\,'F(BX, BZ) + BZ\,'F(BX, BY) \\ &\quad - 'F([BX, BY], BZ) + 'F([BX, BZ], BY) \\ &\quad - 'F([BY, BZ], BX) \\ &= (d\,'F)(BX, BY, BZ) \\ &= 0. \end{aligned}$$

Hence the theorem is proved.

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