ON CODES CORRECTING LEE-BURSTS OF LIMITED INTENSITY WITH DENSITY WITHIN A GIVEN RANGE*

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In most of the communication channels disturbances like lightning, break downs and loose connections affect successive digits for some length of the word, causing errors in burts. It is quite possible that some of the digits within a burst may not be disrupted. This necessitated the study of bursts with weight constraints. Mostly, the work done in this direction uses Hamming weight. This weight is not suitable for phase modulation schemes. For such schemes Lee weight is more appropriate.

In this paper we examine the existence, structure and capabilities of codes detecting/correcting a Lee burst of specified length whose weight lies between two given limits and in which the effect of noise on each digit has a limited intensity.

1. INTRODUCTION

In most of the communication channels disturbances due to lightning, break downs and loose connections affect successive digits for some length of the word, causing errors in bursts. Abramson (1959) initiated the idea of such errors and developed a class of error correcting codes which correct all double adjacent errors. Later, a systematic study in this direction was made by Fire (1959), Rieger (1960) and Elspas (1960). These studies were based on the assumption that if errors occur in the form of bursts then all the digits inside the burst may get disrupted. However, it is quite possible that some of the digits within a burst may not be corrupted. Sharma and Dass (1974) observed this gap and defined bursts with weight constraints. Gupta (1976) further developed these ideas.

All these studies were however restricted to bursts with Hamming weights. As discussed by Berlekemp (1968), in phase modulation schemes, Lee weight is more appropriate than the Hamming weight. So the study of bursts with Lee weights has its own importance in coding theory.

1.1 *Bursts with Lee weight*

Let C be an (n, k) code over a field $Z_q = \{0, 1, 2, ..., q - 1\}$ of integers modulo

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a prime integer q. A vector $(a_1, a_2, \ldots, a_n) \in C$ is called a burst of length b and Lee weight w if all the non-zero entries in this vector are confined to b consecutive positions, the first and the last of which are non-zero and

$$W_L(a_1, a_2, \ldots, a_n) = w$$

where $W_L(a_1, a_2, \ldots, a_n)$ denotes the Lee weight of $(a_1, a_2, \ldots, a_n)$ defined by

$$W_L(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} | a_i |$$

such that

$$| a_i | = \begin{cases} a_i \text{ if } 0 \leqslant a_i \leqslant \frac{1}{2} q, \\ - a_i \pmod q \text{ if } \frac{1}{2} q < a_i \leqslant q - 1. \end{cases}$$

In this paper, whenever we talk of weight, we shall mean it in Lee sense only. Also Bursts with Lee weight will be called "Lee bursts".

Since for $q = 2, 3$ the Hamming and Lee considerations coincide, therefore in studing a linear code for Lee burst errors, we will restrict to codes over a field Z_q, where q is an odd prime ($q \geqslant 3$).

We may also emphasize here that the field elements a_i and $q - a_i$ are equivalent for the purpose of determining the weight. There may be a further restriction on the errors that may occur. This can be in terms of the magnitude of digital change in a position. So we may assume that the channel is such that no digit of the code word is changed by more than a number say α. Obviously $\alpha \leqslant \left(\frac{q-1}{2}\right)$. We shall call α as the intensity of the noise (or error). Without any loss of generality we may consider that each non-zero entry in a vector is less than or equal to α with two equivalent repetitive values.

We shall denote the largest integral part of x by $[x]$ and the least integer greater than or equal to x by $\{x\}$.

Our aim in this paper is to examine the structure and capabilities of codes detecting/correcting bursts of specified length with weights lying between two given limits and in which the noise has a limited intensity.

In our studies we extensively use the combinatorial results concerning the filling up of certain number of positions by integers from the set $\{0, 1, 2, \ldots, \alpha\}$ to make up a specified sum (refer Berge 1971 and Goel 1976 for details).

2. Bursts of Specified Length with Weights Lying between two given Limits

Theorem 2.1 — Let us denote by $V_n(Z_q)$ the vector space of all n-tuples over the field Z_q of integers modulo a prime odd integer q. If $B_L(n, b, w_1, w_2)$ denotes the number of n-vectors forming, Lee bursts of length b or less with weight lying between w_1 and w_2 $(0 \leqslant w_1 \leqslant w_2 \leqslant \alpha b)$, then

$$B_L(n, b, w_1, w_2) = \mu + \sum_{m=i}^{b} (n - m + 1)$$

$$\times \sum_{\eta = w_1 \geqslant 2}^{w_2} \sum_{\lambda_2}^{\min(\eta-\lambda_1,\alpha)} \sum_{\lambda_1}^{\min(\eta-1,\alpha)} \cdot \sum_{k=\{\eta-\lambda_1-\lambda_2/m-2\}}^{\min(\eta-\lambda_1-\lambda_2,\alpha)} A^k(\eta - \lambda_1 - \lambda_2, m-2)\, 2^2 \qquad \ldots(2.1)$$

where $A^k(\eta - \lambda_1 - \lambda_2, m - 2)$* denotes the number of arrangements (each having the largest entry k) of filling $(m - 2)$ positions with integers from the set $\{0, 1, 2, \ldots, \alpha\}$ with each non-zero entry having 2 equivalent repetitive values, to make up the sum $\eta - \lambda_1 - \lambda_2$. And μ and i are given by the relations

$$\mu = \begin{cases} 1 + 2n \, . \min(\alpha, w_2) & \text{if } w_1 = 0, \\ 2n \min(\alpha, w_2) & \text{if } 1 \leqslant w_1 \leqslant \alpha, \\ 0 & \text{if } w_1 > \alpha. \end{cases} \qquad \ldots(2.2)$$

and $$\left\{\frac{w_1}{i}\right\} \leqslant (\alpha) < \left\{\frac{w_1}{i-1}\right\}.$$

PROOF: It is clear that there is only 1 burst of length zero, namely the null vector. Also the number of bursts of length 1 with weight lying between w_1 and w_2 is $2n \, . \min(\alpha, w_2)$ or 0 according as $1 \leqslant w_1 \leqslant \alpha$ or $w_1 > \alpha$.

Starting from a fixed position of the n-vectors, let us consider the number of bursts of length say $m(2 \leqslant m \leqslant b)$ with weight say η. Evidently, each of such bursts has at least 2 non-zero entries, let one of them contribute a weight λ_1 and another the weight λ_2. Clearly, $1 \leqslant \lambda_1 \leqslant \min(\eta - 1, \alpha)$ and for a fixed λ_1 we have,

$$1 \leqslant \lambda_2 \leqslant \min(\eta - \lambda_1, \alpha).$$

Since each non-zero entry (including λ_1 and λ_2) in the burst has 2 equivalent repetitive values, so the number of ways of obtaining such a burst is

$$A^k(\eta - \lambda_1 - \lambda_2, m - 2)\, 2^2.$$

$${}^*A^k(\eta - \lambda_1 - \lambda_2, m - 2) = \sum_{t_0, t_1, \ldots, t_k} \frac{m - 2!}{t_0!\, t_1! \ldots t_k!} 2^{t_1 + \cdots + t_k}$$

where t_i denotes the number of times the integer i occurs and therefore t_i's are non-negative integers such that

$$\sum_{i=0}^{k} t_i = m - 2,\ t_k \geqslant 1 \text{ and } \sum_{i=1}^{k} i \, . \, t_i = \eta - \lambda_1 - \lambda_2.$$

Clearly, $\left\{\frac{\eta - \lambda_1 - \lambda_2}{m - 2}\right\} \leqslant k \leqslant \min(\eta - \lambda_1 - \lambda_2, \alpha)$.

Moreover, variations of λ_1, λ_2 and k give different bursts. Thus the total number of such bursts of length m and weight η is given by

$$\sum_{\lambda_2} \sum_{\lambda_1} \sum_{k} A^k(\eta - \lambda_1 - \lambda_2, m - 2)\, 2^2 \qquad \ldots(2.3)$$

It is clear that each such burst has weight at least 2, since $m \geqslant 2$. Now allowing η to vary from $w_1(\geqslant 2)$ to w_2 summation of (2.3) gives the total number of all those bursts of length m with weight lying between w_1 and w_2 which start from the initially fixed position.

Since a burst of length m over a vector of length n can have $n - m + 1$ starting positions and since $i \leqslant m \leqslant b$ where

$$\left\{\frac{w_1}{i}\right\} \leqslant \alpha < \left\{\frac{w_1}{i - 1}\right\},$$

the total number of bursts of length b or less with weight lying between w_1 and w_2 is therefore, given by (2.1)

Corollary 2.2 — Putting $w_1 = 0$ and $w_2 = w$ in (2.1) we obtain the total number of bursts of length b or less with weight w or less and is given by

$$B_L(n, b, w) = 1 + 2n \min(\alpha, w) + \sum_{m} (n - m + 1) \sum_{\eta} \sum_{\lambda_2} \sum_{\lambda_1} \sum_{k} A^k(\eta - \lambda_1 - \lambda_2, m - 2)\, 2^2.$$

Theorem 2.3 — A sufficient condition for the existence of an (n, k) linear code that has no burst of length b or less with weight lying between w_1 and w_2, as a code word is

$$q^{n-k} \geqslant \mu + \sum_{\eta = w_1 \geqslant 2}^{w_2} \; \sum_{\lambda=1}^{\min(\eta-1,\alpha)} \; \sum_{k=\{\eta-\lambda/b-1\}}^{\min(\eta-\lambda-b+2,\alpha)} A^k(\eta - \lambda, b - 1)\, 2 \qquad \ldots(2.4)$$

where

$$\mu = \begin{cases} 1 + n & \text{when } w_1 = 0, \\ n & \text{when } w_1 = 1, \\ 0 & \text{when } w_1 > 1. \end{cases}$$

PROOF: The condition for the existence of such a linear code may be obtained by examining the possibility of constructing an $(n - k) \times n$ parity check matrix for the desired code.

We may choose an arbitrary non-zero $(n - k)$-tuple h_1 as the first column of H. Subsequent columns are added in such a way that after having selected first $(j - 1)$ columns, a non-zero $(n - k)$-tuple h_j may be added as jth column if any linear combination of h_j with $b - 1$ or less immediately preceding columns taken in such a way that the coefficient vector of all these columns (including h_j) contribute a weight lying between w_1 and w_2, should not vanish.

Thus H can be completed if this process continues upto $j = n$, i.e. if a non-zero $(n - k)$-tuple h_n is available such that

$$a_n h_n + a_{n-1} h_{n-1} + \dots + a_{n-(b-1)} h_{n-(b-1)} \neq 0 \qquad \dots(2.5)$$

whenever

$$w_1 \leqslant w_L(a_n, a_{n-1}, \dots, a_{n-(b-1)}) \leqslant w_2,\ a_i\text{'s} \in Z_q.$$

Thus the possible number of ways of selecting h_j's satisfying (2.5) is the same as the number of ways of selecting the sets of a_i's $\in Z_q$ (not exceeding b in each set) such that the weight contributed by each set lies between w_1 and w_2.

This number is clearly,

$$\mu + \sum_{\eta = w_1 \geqslant 2}^{w_2} \ \sum_{\lambda=1}^{\min(\eta-1,\alpha)} \ \sum_{k=\{\eta-\lambda/b-1\}}^{\min(\eta-\lambda-b+2,\alpha)} A^k(\eta - \lambda, b - 1)\, 2$$

where

$$\mu = \begin{cases} 1 + n & \text{when } w_1 = 0, \\ n & \text{when } w_1 = 1, \\ 0 & \text{when } w_1 > 1. \end{cases}$$

Since this number should not exhaust all $(n - k)$-tuples, so we obtain condition (2.4).

This is essentially the condition required for the existence of an (n, k) code detecting the above mentioned type of burst errors. Now we derive a necessary and sufficient condition for the existence of a code correcting such type of errors.

3. Bounds Over the Number of Parity Check Digits of a Linear Code Correcting Bursts of Specified Length with their Weights between two given Limits

Theorem 3.1 — The number of parity check digits for an (n, k) linear code that corrects all bursts of length b or less with weight lying between w_1 and w_2 $(0 \leqslant w_1 \leqslant w_2 \leqslant \alpha b)$ is at least

$$\log_q [c + B_L(n, b, w_1, w_2)] \qquad ...(3.1)$$

where

$$c = \begin{cases} 0 & \text{if } w_1 = 0, \\ 1 & \text{otherwise.} \end{cases}$$

and $B_L(n, b, w_1, w_2)$ is given by (2.1).

PROOF : As discussed earlier $B_L(n, b, w_1, w_2)$ contains the null vector only when $w_1 = 0$. So the number of correctable patterns, including the pattern of all zeros, is

$$c + B_L(n, b, w_1, w_2)$$

where

$$c = \begin{cases} 0 & \text{when } w_1 = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Now, since the code is required to correct all bursts of length b or less with weights lying between w_1 and w_2, so the number of all error patterns must be in different cosets. Consequently, $q^{n-k} \geqslant c + B_L(n, b, w_1, w_2)$

Taking logarithms to the base q, result (3.1) is obtained.

In the next theorem, we derive a sufficient condition for the existence of an (n, k) linear code which is capable of correcting bursts of length b or less with weight lying between w_1 and w_2. For such a code, no code vector should be expressible as a difference of two bursts of length b or less with weight lying between w_1 and w_2, and no code vector should be a burst of length b or less with weight lying between w_1 and w_2. Consequently, if H is the parity check matrix of the required code then (i) no linear combination of any b consecutive columns of H, with their coefficient vector having the weight lying between w_1 and w_2, should vanish and (ii) no linear combination involving two sets of b or less consecutive columns of H should be zero, if in each set the coefficient vector has a weight lying between w_1 and w_2.

Now we construct H in accordance with the above rule. In the process of our construction, first $(i - 1)$ columns may be chosen by taking arbitrary non-zero $(n - k)$-tuples, where

$$\left\{\frac{w_1}{i}\right\} \leqslant \alpha < \left\{\frac{w_1}{i-1}\right\}.$$

Now subsequent columns may be added in such a way that after having selected $(j - 1)$ columns $(j > i - 1)$ appropriately, an $(n - k)$-tuple h_j may be chosen as the jth column of H, if

$$\left.\begin{aligned} a_j h_j \neq (a_1 h_{j-1} + a_2 h_{j-2} + \ldots + a_{b-1} h_{j-b+1}) \\ + (c_k h_k + c_{k+1} h_{k+1} + \ldots + c_{k+b-1} h_{k+b-1}) \end{aligned}\right\} \quad \ldots(3.2)$$

where a_i's, c_k's are in Z_q and h_k's are any b consecutive columns from all the $(j-1)$ columns chosen in such a way that,

$$w_1 \leqslant w_L(a_1, a_2, \ldots, a_{b-1}, a_j) \leqslant w_2$$

and

$$w_1 \leqslant w_L(c_k, c_{k+1}, \ldots, c_{k+b-1}) \leqslant w_2$$

Now all possible linear combinations corresponding to (3.2) for all possible choices of a_i's and c_k's can be computed by analysing the following three cases :

Case I — When the h_k's are chosen amongst $h_{j-b+1}, \ldots, h_{j-1}$ columns

Since $a_j \neq 0$, let it contribute the weight λ. Then the possible number of ways of selecting a_i's and c_k's satisfying (3.2) is given by

$$C_1 = \sum_{\eta=1}^{2w_2} \sum_{\lambda=1}^{\min(\eta-1,\alpha)} \sum_{k=\{\eta-\lambda/b-1\}}^{\min(\eta-\lambda-(b-2),\alpha)} A^k(\eta-\lambda, b-1)\,2 \quad \ldots(3.3)$$

Case II — When the h_k's are chosen amongst $h_1, h_2, \ldots, h_{j-b}$ columns

In this case the number of linear combinations corresponding to (3.2) and that have not been accounted for in Case I, is

$$C_2 = B_L(j-b, b^*, w, w_2) \,.\, B_L(b-1, b-1, w_1-\alpha, w_2-1) \quad \ldots(3.4)$$

where $b^* = \min(j-b, b)$ and $w = \max(1, w_1)$

and $B_L(n, b, w_1, w_2)$ is given by (2.1).

Case III — When the h_k's are chosen partly from amongst $h_1, h_2, \ldots, h_{j-b}$ and partly from $h_{j-b+1}, \ldots, h_{j-1}$.

Since a burst of length b or less formed by the corresponding coefficients of h_k's, in this case can start only from any of the positions from $(j-2b+2)$th to $(j-b)$th positions, let it start from the $(j-2b+2+k)$th position $(0 \leqslant k \leqslant b-2)$. Then it may go up to $(j-b+1+k)$th position.

Let r_1, r_2 and r_3 be the number of non-zero positions of the burst formed by the corresponding positions of h_k's and the h_i's in (3.2) chosen amongst $(j-2b+2+k+1)$th to $(j-b)$th, $(j-b+1)$th to $(j-b+k+1)$th and $(j-b+k+2)$th to $(j-1)$th positions respectively. Then all possible number of choices of a's and c_k's satisfying (3.2) is given by

$$C_3 = \sum_{k=0}^{b-2} \sum_{\eta_1,\eta_2,\eta_3} \sum_{r_1,r_2,r_3} \Big[\Big\{ \binom{b-k-2}{r_1} \sum_{\eta_1} \sum_{k_1} A^{k_1}(\eta_1, r_1) \Big\}$$

$$\Big\{ \binom{k+1}{r_2} \sum_{\eta_2} \sum_{k_2} A^{k_2}(\eta_2, r_2) \Big\} \Big\{ \binom{b-k-1}{r_3} \sum_{\eta_3} \sum_{k_3} A^{k_3}(\eta_3, r_3) \Big\} \Big] \qquad \ldots(3.5)$$

where

$$\left\{ \frac{\eta_i}{r_i} \right\} \leqslant k_i \leqslant \min \{\eta_i - 1, \alpha\}$$

and η_1, η_2, η_3 are the weights contributed by r_1, r_2 and r_3 positions respectively governed by the following conditions.

$$0 \leqslant r_1 \leqslant b - 2 \qquad w \leqslant \eta_1 \leqslant w_2 - 2$$

$$0 \leqslant r_3 \leqslant b - 1 \qquad w \leqslant \eta_3 \leqslant w_2 - 1$$

$$2 \leqslant r_1 + r_2 \leqslant b - 1 \qquad w_1 - \alpha \leqslant \eta_1 + \eta_2 \leqslant w_2 - 1$$

$$r_1 + r_2 + r_3 \leqslant 2b - 2 \qquad \eta_1 + \eta_2 + \eta_3 \leqslant 2w_2 - 2$$

$$1 \leqslant r_2 + r_3 \leqslant b - 1 \qquad w_1 - \alpha \leqslant \eta_2 + \eta_3 \leqslant 2w_2 - 1$$

where

$$w = \max . \{0, w_1 - \alpha - \alpha(k + 1)\}$$

Thus the total number of linear combinations corresponding to (3.2) is given by

$$C_1 + B_L(j - b, \min \{j - b, b\}, \max . \{1, w_1\}, w_2)$$
$$\times B_L(b - 1, b - 1, w_1 - \alpha, w_2 - 1) + C_3 \qquad \ldots(3.6)$$

where C_1 and C_3 are given by (3.3) and (3.5) respectively. Thus, jth column can be added if the expression given in (3.6) does not exhaust all $(n - k)$-tuples.

Now H can be completed if the expression in (3.6) for $j = n$ does not exceed q^{n-k}.

Thus it proves the following theorem :

Theorem 3.2 — A sufficient condition for the existence of an (n, k) linear code which corrects all bursts of length b or less with weight lying between w_1 and $w_2(0 \leqslant w_1 \leqslant w_2 \leqslant \alpha b)$, is

$$q^{n-k} \geqslant C_1 + B_L(n - b, \min \{n - b, b\}, \max \{1, w_1\})$$
$$\times B_L(b - 1, b - 1, w_1 - \alpha, w_2 - 1) + C_3$$

where C_1 and C_3 are given by (3.3) and (3.5) respectively and $B_L(n, b, w_1, w_2)$ is given by (2.1).

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