ON SOME TRANSFORMATIONS OF TRIPLE HYPERGEOMETRIC SERIES $F^{(3)}$—III

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This paper is the third in a series. The first two papers are devoted to the study of the linear transformations of a general triple hypergeometric series $F^{(3)}$ of Srivastava (1967). This paper studies a reduction formula of $F^{(3)}$ into a combination of Kampé de Fériet's function $F^{(2)}$, together with its special cases.

1. INTRODUCTION

In earlier papers (Pathan 1978a, b) appear some linear transformations and reduction formulas of a general triple hypergeometric series $F^{(3)}$ introduced by Srivastava (1967, p. 428). This paper deals with a reduction formula of $F^{(3)}$ into a combinations of Kampé de Fériet's function $F^{(2)}$ in the following form :

$$(1-x)^{2a} F^{(3)}\left[\begin{matrix} 2a :: 2b; -; - : d; d-2e+1; 2c-2b; \\ 2c :: -; -; - : 2e; \quad 2-2e \quad ; \quad - \quad ; \end{matrix} \; x, x, y\right]$$

$$= F^{(2)}\left[\begin{matrix} a, a+\tfrac{1}{2} : \tfrac{1}{2}+d-e, \tfrac{1}{2}-d+e, b, b+\tfrac{1}{2}; c-b, c-b+\tfrac{1}{2}; \\ c, c+\tfrac{1}{2} : \quad \tfrac{1}{2}, \tfrac{1}{2}+e, \quad \tfrac{3}{2}-e \quad ; \quad \tfrac{1}{2} \quad ; \end{matrix} \; u^2, v^2\right]$$

$$+ \frac{2a(c-b)\,v}{c}$$

$$\times F^{(2)}\left[\begin{matrix} a+\tfrac{1}{2}, a+1 : \tfrac{1}{2}+e-d, \tfrac{1}{2}-e+d, b, b+\tfrac{1}{2}; c-b+\tfrac{1}{2}, c-b+1; \\ c+\tfrac{1}{2}, c+1 : \quad \tfrac{1}{2}, \quad \tfrac{1}{2}+e, \tfrac{3}{2}-e \quad ; \quad \tfrac{3}{2} \quad ; \end{matrix} \; u^2, v^2\right]$$

$$+ \frac{abu(2e-1)(e-d)}{2e(1-e)\,c}$$

$$\times \left\{ F^{(2)}\left[\begin{matrix} a+\tfrac{1}{2}, a+1 : 1+e-d, 1-e+d, b+\tfrac{1}{2}, b+1; c-b, c-b+\tfrac{1}{2}; \\ c+\tfrac{1}{2}, c+1 : \quad \tfrac{3}{2} \quad, 1+e \quad, \quad 2-e \quad ; \quad \tfrac{1}{2} \quad ; \end{matrix} \; u^2, v^2\right] + \right.$$

(*equation continued on p.* 1114)

$$+\frac{(2a+1)(c-b)v}{(c+\frac{1}{2})}$$

$$\times F^{(2)}\left[\begin{matrix} a+1, a+\frac{3}{2}: 1+e-d, 1-e+d, b+\frac{1}{2}, b+1; c-b+\frac{1}{2}, c-b+1; \\ c+1, c+\frac{3}{2}: \quad \frac{3}{2} \quad, \quad 1+e \quad, \quad 2-e \quad; \quad \frac{3}{2} \quad; \end{matrix} u^2, v^2\right]\Bigg\} \qquad \ldots(1.1)$$

where $u = \dfrac{x}{(1-x)}$ and $v = \dfrac{y-x}{(1-x)}$.

The notations and definitions used in this paper are the same as in Pathan (1978a).

Our approach in formulating (1.1) is based on the use of certain techniques discussed in earlier two papers (Pathan 1978a, b). Although the method has some features in common with those of Pathan (1978a, b), it will nevertheless yield sharper results in different situations by specializing the parameters or variables or both. As particular cases of our result we will obtain in §3 some reduction formulae for Kampé de Fériet's function $F^{(2)}$.

2. Derivation of Formula (1.1)

In a known result of the author (Pathan 1978a)

$$\int_0^\infty t^\lambda \exp\left[-z-\tfrac{1}{2}(p-x-y)\right] t\, W_{K,\mu}(pt)\, M_{\sigma,\nu}(xt)\, M_{\eta,\xi}(yt)\, dt$$

$$= \frac{x^{\nu+(1/2)}\, y^{\xi+(1/2)}\, p^{\mu+(1/2)}\, \Gamma(a+\mu)\,\Gamma(a-\mu)}{(z+p)^{a+\mu}\,\Gamma(a-K+\frac{1}{2})}$$

$$\times F^{(3)}\left[\begin{matrix} a+\mu \;::\; a-\mu; -; -: \frac{1}{2}-\sigma+\nu; \frac{1}{2}-\eta+\xi; \mu-K+\frac{1}{2}; \\ a-K+\frac{1}{2} :: \; - \; ; -; -: \; 2\nu+1 \; ; \; 2\xi+1 \; ; \; - \; ; \end{matrix} \frac{x}{z+p}, \frac{y}{z+p}, \frac{z}{z+p}\right] \qquad \ldots(2.1)$$

where $a = \lambda+\nu+\xi+\frac{5}{2}$, $\mathrm{Re}(a+\mu) > 0$ and $\mathrm{Re}(2z+p-x-y \pm p \pm x \pm y) > 0$, expressing $M_{\sigma,\nu}(xt)$ in terms of ${}_1F_1$, using

$$M_{\eta,\xi}(yt) = (yt)^{\xi+(1/2)}\, e^{(1/2)yt}\, {}_1F_1\left(\tfrac{1}{2}+\eta+\xi, 2\xi+1; -yt\right) \qquad \ldots(2.2)$$

in conjugation with a result of MacRobert (1962, p. 395)

$${}_1F_1(\alpha, \rho; x)\, {}_1F_1(\alpha-\rho+1, 2-\rho; -x) = {}_2F_3\left(\begin{matrix} \frac{1}{2}+\frac{1}{2}\rho-\alpha, \frac{1}{2}-\frac{1}{2}\rho+\alpha \\ \frac{1}{2}, \frac{1}{2}+\frac{1}{2}\rho, \frac{3}{2}-\frac{1}{2}\rho \end{matrix}; x^2\right)$$

$$+\frac{(\rho-1)(\rho-2\alpha)}{\rho(2-\rho)}\, x\, {}_2F_3\left(\begin{matrix} 1+\frac{1}{2}\rho-\alpha, 1-\frac{1}{2}\rho+\alpha \\ \frac{3}{2}, 1+\frac{1}{2}\rho, 2-\frac{1}{2}\rho \end{matrix}; x^2\right), \qquad \ldots(2.3)$$

expanding ${}_2F_3$'s in series integrating term by term with the help of the result of Erdeyli *et al.* [1954, p. 216(16)], we arrive at an expression in which ${}_2F_1$ appears. Again, applying a result of Carlson (1970, p. 234)

$$
{}_2F_1\left(\begin{matrix} a, b \\ \\ c \end{matrix}; x\right) = {}_4F_3\left(\begin{matrix} \tfrac{1}{2}a, \tfrac{1}{2}a + \tfrac{1}{2}, \tfrac{1}{2}b, \tfrac{1}{2}b + \tfrac{1}{2} \\ \\ \tfrac{1}{2}, \tfrac{1}{2}c, \tfrac{1}{2}c + \tfrac{1}{2} \end{matrix}; x^2\right)
$$

$$
+ \frac{ab}{c} x\, {}_4F_3\left(\begin{matrix} \tfrac{1}{2}a + \tfrac{1}{2}, \tfrac{1}{2}a + 1, \tfrac{1}{2}b + \tfrac{1}{2}, \tfrac{1}{2}b + 1 \\ \\ \tfrac{3}{2}, \tfrac{1}{2}c + \tfrac{1}{2}, \tfrac{1}{2}c + 1 \end{matrix}; x^2\right), \qquad \ldots(2.4)
$$

expressing ${}_4F_3$ in series and interpreting the result in terms of $F^{(3)}$ would yield our transformation formula (1.1).

3. Special Cases

Setting $y = x$ in (1.1) and using a result of the author (Pathan 1978a)

$$
F^{(3)}\left[\begin{matrix} a :: b; -; - : d; g; c - b; \\ \\ c :: -; -; - : e; f; \;\; - \;\; ; \end{matrix} \;\; \frac{x}{1+x}, \frac{x}{1+x}, \frac{x}{1+x}\right]
$$

$$
= (1 + x)^{a} F^{(2)}\left[\begin{matrix} a, b : g; e - d; \\ \\ c \;\; : f; \;\; e \;\; ; \end{matrix} \;\; x, -x\right], \qquad \ldots(3.1)
$$

we get a reduction formula for Kampé de Fériet's function $F^{(2)}$ in the form

$$
F^{(2)}\left[\begin{matrix} 2a, 2b : d - 2e + 1; 1 - e; \\ \\ 2c \;\; : \;\; 2 - 2e \;\; ; \;\; 2e \;\; ; \end{matrix} \;\; x, -x\right]
$$

$$
= {}_6F_5\left(\begin{matrix} a, a + \tfrac{1}{2}, b, b + \tfrac{1}{2}, \tfrac{1}{2} + e - d, \tfrac{1}{2} - e + d \\ \\ c, c + \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} + e, \;\; \tfrac{3}{2} - e \end{matrix}; x^2\right)
$$

$$
+ \frac{(2e - 1)(e - d)\, abx}{2e(1 - e)\, c}
$$

$$
\times\, {}_6F_5\left(\begin{matrix} a + \tfrac{1}{2}, a + 1, b + \tfrac{1}{2}, b + 1, 1 + e - d, 1 - e + d \\ \\ c + \tfrac{1}{2}, c + 1, \;\; \tfrac{3}{2} \;\; , 1 + e, \;\; 2 - e \end{matrix}; x^2\right) \qquad \ldots(3.2)
$$

which for $b = c$ reduces to

$$F_2(2a, d - 2e + 1, 1 - d, 2 - 2e, 2e; x, -x)$$

$$= {}_4F_3\left(\begin{matrix} a, a + \frac{1}{2}, \frac{1}{2} + d - e, \frac{1}{2} - d + e \\ \frac{1}{2}, \frac{1}{2} + e, \quad \frac{3}{2} - e \end{matrix}; x^2\right)$$

$$+ \frac{(2e - 1)(e - d)\, ax}{2e(1 - e)}$$

$$\times\, {}_4F_3\left(\begin{matrix} a + \frac{1}{2}, a + 1, 1 + e - d, 1 - e + d \\ \frac{3}{2}, 1 + e, \quad 2 - e \end{matrix}; x^2\right) \qquad ...(3.3)$$

where F_2 is Appell's function of second kind.

Specialization of (3.2) by putting $d = e$ gives a reduction formula

$$F^{(2)}\left[\begin{matrix} 2a, 2b : 1 - d ; 1 - d; \\ 2c \; : 2 - 2d; \; 2d \; ; \end{matrix} x, -x\right]$$

$$= {}_5F_4\left(\begin{matrix} a, a + \frac{1}{2}, \; \frac{1}{2}, \; b, b + \frac{1}{2} \\ c, c + \frac{1}{2}, \frac{1}{2} + d, \; \frac{3}{2} - d \end{matrix}; x^2\right). \qquad ...(3.4)$$

When $y \to 0$, $F^{(3)}$ in (1.1) reduces to $F^{(2)}$ and the right hand side yields a combination of four $F^{(2)}$'s. On the other hand, if we take $d = e$ then we are led to a result

$$(1 - x)^{2a} F^{(3)}\left[\begin{matrix} 2a :: 2b; -; - : d ; 1 - 2d; 2c - 2b; \\ 2c :: -; -; - : 2d; 2 - 2d; \quad - \quad ; \end{matrix} x, x, y\right]$$

$$= F^{(2)}\left[\begin{matrix} a, a + \frac{1}{2} : \frac{1}{2}, \; b \;, b + \frac{1}{2}; c - b, c - b + \frac{1}{2}; \\ c, c + \frac{1}{2} : \frac{1}{2} + d, \frac{3}{2} - d; \quad \frac{1}{2} \quad ; \end{matrix} u^2, v^2\right]$$

$$+ \frac{2a(c - b)\, v}{c}$$

$$\times\, F^{(2)}\left[\begin{matrix} a + \frac{1}{2}, a + 1 : \frac{1}{2}, \; b \;, b + \frac{1}{2}; c - b + \frac{1}{2}, c - b + 1; \\ c + \frac{1}{2}, c + 1 : \frac{1}{2} + d, \frac{3}{2} - d; \quad \frac{3}{2} \quad ; \end{matrix} u^2, v^2\right], \qquad ...(3.5)$$

where $\quad u = \dfrac{x}{1 - x}$ and $v = \dfrac{y - x}{1 - x}$.

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