ON THE UNIVALENCE OF SOME ANALYTIC FUNCTIONS—II

by V. N. KULKARNI *and* S. R. SWAMY*, *Department of Mathematics, Karnatak University, Dharwad* 580003

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Let $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots, g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \ldots,$ and $h(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots$ be analytic and satisfy

(a) $$\left| \frac{zf(z)}{\lambda zf(z) + (1 - \lambda)\, g(z)\, h(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta}$$

for $|z| < 1$, $0 \leqslant \lambda < 1$, $0 \leqslant \beta < 1$, where the case $\beta = 0$ corresponds to

(b) $$\operatorname{Re}\left\{\frac{zf(z)}{\lambda zf(z) + (1 - \lambda)\, g(z)\, h(z)}\right\} > 0.$$

We propose to determine the values of R_t such that $f(z)$ is univalent and starlike for $|z| < R_t^{1/n}$ under the assumption

(i) $$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \alpha,\ 0 \leqslant \alpha \leqslant 1, \text{ and } \operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} > \gamma, 0 \leqslant \gamma \leqslant 1$$

or,

(ii) $$\operatorname{Re}\left\{\frac{g(z)}{u(z)}\right\} > \delta \text{ or } \left|\frac{g(z)}{u(z)} - \frac{1}{2\delta}\right| < \frac{1}{2\delta},\ 0 \leqslant \delta < 1,$$

where

$$\operatorname{Re}\left\{\frac{zu'(z)}{u(z)}\right\} > \alpha,\ 0 \leqslant \alpha \leqslant 1, \text{ and } \operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} > \gamma,\ 0 \leqslant \gamma \leqslant 1.$$

§1. Let S denote the class of all analytic functions $f(z)$ defined in the unit disc and normalized by the conditions $f(0) = f'(0) - 1 = 0$. For a fixed α, $0 \leqslant \alpha \leqslant 1$, let $S^*(\alpha)$ denote the class consisting of all functions $f(z) \in S$ satisfying the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \text{ for } |z| < 1.$$

Functions in the class $S^*(\alpha)$ are known as starlike functions of order α. If $\alpha = 1$, then $f(z) = z$.

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In this paper conditions for starlikeness for subclasses of the class S are developed. Let $p(\beta, n)$ denote the class of functions analytic in $|z| < 1$, with expansion $p(z) = 1 + d_n z^n + d_{n+1} z^{n+1} + \ldots, n \geqslant 1$, which satisfy the inequality

$$\left| p(z) - \frac{1}{2\beta} \right| < \frac{1}{2\beta},\ 0 \leqslant \beta < 1, \text{ for } |z| < 1,$$

and $H(\beta, n)$ the class satisfying

$$\text{Re}\,(p(z)) > \beta,\ 0 \leqslant \beta < 1, \text{ for } |z| < 1.$$

If $\beta = 0$ both the classes $p(\beta, n)$ and $H(\beta, n)$ reduce to the class E of functions with positive real part. New results for the class S are obtained from inequalities for $p(\beta, n)$ and $H(\beta, n)$.

§2. We need the following Lemmas for our discussion. Following notations $c = 1 - 2\beta$, $d = 2\alpha - 1$, $e = 2\gamma - 1$, and $m = 1 - 2\delta$ are used in this paper.

Lemma 1 (Shah 1972, Lemma 2) — Let $p(z) \in H(\beta, n)$, then we have for $|z| < 1$.

$$\left| \frac{zp'(z)}{p(z)} \right| \leqslant \frac{(1 + c)\, n\, |z|^n}{(1 + c|z|^n)(1 - |z|^n)}.$$

Lemma 2 (Shah 1972, Lemma 3) — Let $p(z) \in H(\alpha, n)$, then we have for $|z| < 1$

$$\text{Re}\,(p(z)) \geqslant \frac{1 + d|z|^n}{1 + |z|^n}.$$

Lemma 3 (Chen 1974, Lemma 4) — Let $p(z) \in p(\beta, n)$, then we have for $|z| < 1$.

$$\left| \frac{zp'(z)}{p(z)} \right| \leqslant \frac{(1 + c)\, n\, |z|^n}{(1 + c|z|^n)(1 - |z|^n)}.$$

Lemma 4 (Chen 1974, Lemma 5) — Let $p(z) \in p(\beta, n)$. For any λ with $0 < \lambda < 1$ and $c + \lambda > 0$, we have for $|z| < \min\left\{1, \left(\frac{1 - \lambda}{c + \lambda}\right)^{1/n}\right\}$

$$|1 - \lambda p(z)|^{-1} \leqslant \frac{1 - c|z|^n}{[(1 - c|z|^n) - \lambda(1 + |z|^n)]}.$$

§3. Now we prove the following results. In all of the following theorems $f(z)$, $g(z)$, and $h(z)$ are normalized analytic functions in the unit disc.

Theorem 1 — Let $g(z) \in S^*(\alpha)$ and $h(z) \in S^*(\gamma)$. If $zf(z)/[\lambda z f(z) + (1 - \lambda) \times g(z)\, h(z)] \in p(\beta, n)$ and $c + \lambda > 0$ whenever $\lambda > 0$. Then $f(z)$ is univalent and starlike for $|z| < R_t^{1/n}$, where R_t is the smallest positive root of the equation

$$H(R) = xc(c+\lambda) R^4 + (nc^2 + nc + c^2 - xc^2 + x\lambda + c\lambda) R^3$$
$$+ (nc^2 - n - c^2 - x + \lambda - xc\lambda) R^2$$
$$+ (x - 1 - n - nc - x\lambda - c\lambda) R + (1-\lambda) = 0 \qquad ...(1)$$

where $x = d + e - 1$.

PROOF: Let

$$p(z) = zf(z)/[\lambda zf(z) + (1-\lambda) g(z) h(z)]. \qquad ...(2)$$

Then $p(z)$ is analytic and $p(z) \in p(\beta, n)$. From (2) we obtain

$$[1 - \lambda p(z)] zf(z) = (1-\lambda) g(z) h(z) p(z). \qquad ...(3)$$

Multiplying the logarithmic derivative of both sides of (3) by z we get

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} + \frac{zp'(z)/p(z)}{1-\lambda p(z)} - 1 \qquad ...(4)$$

and this gives

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \text{Re}\left\{\frac{zg'(z)}{g(z)}\right\} + \text{Re}\left\{\frac{zh'(z)}{h(z)}\right\} - 1 - \left|\frac{zp'(z)/p(z)}{1-\lambda p(z)}\right|. \qquad ...(5)$$

Applying Lemma 2, Lemma 3, and Lemma 4 we get

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{1+dR}{1+R} + \frac{1+eR}{1+R} - 1$$
$$- \frac{(1+c) nR(1-cR)}{(1+cR)(1-R)[(1-cR) - \lambda(1+R)]}$$
$$= \frac{1+xR}{1+R} - \frac{(1+c) nR(1-cR)}{(1+cR)(1-R)[(1-\lambda)-(c+\lambda) R]} \qquad ...(6)$$

provided $|z| < \min\left\{1, \left(\frac{1-\lambda}{c+\lambda}\right)^{1/n}\right\}$, where $R = |z|^n$ and $x = d + e - 1$. Hence $\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ if $|z| < \min\left\{1, \left(\frac{1-\lambda}{c+\lambda}\right)^{1/n}\right\}$ and if right-hand side of (6) is greater than zero. The latter will hold if $H(R) > 0$. Since $H(0) = 1 - \lambda > 0$, $H(1) = 2n(c^2 - 1) \leqslant 0$, and $H[(1-\lambda)/(c+\lambda)] = \frac{-n\lambda(1-\lambda)(1+c)^3}{(c+\lambda)^3} < 0$ it follows that $H(R) = 0$ has at least one root in $(0, M]$, where $M = \min\left\{1, \frac{1-\lambda}{c+\lambda}\right\}$.

Let R_t be the smallest positive root of $H(R) = 0$. Then $0 < R_t \leqslant M$ and $H(R) > 0$ for $0 \leqslant R < R_t$. That is $f(z)$ is univalent and starlike for $|z| < R_t^{1/n}$. This completes the proof of Theorem 1.

Corollary 1.1 — Let $\alpha = \gamma = 1$. Then we obtain Theorem 1 of Chen (1974) which, when $\lambda = 0$ reduces to a result of Chichra (1972, Theorem 2).

PROOF : Put $x = 1$ in (1), then we get

$$H(R) = (R + 1)\, H_1(R)$$

where

$$H_1(R) = c(c + \lambda)\, R^3 + (nc^2 + nc - c^2 + \lambda)\, R^2 - (nc + n + 1 + c\lambda)\, R + (1 - \lambda).$$

Corollary 1.2 — Let $\beta = 0$. Then we obtain a result of Swamy (1978, Theorem 2) which, when $\gamma = 1$ reduces to a result of Shah (1972, Theorem 3).

PROOF : Put $c = 1$ in (1), then we get

$$H(R) = (R - 1)\,(R + 1)\, H_2(R)$$

where

$$H_2(R) = (1 + \lambda)\, xR^2 + (1 + 2n - x + \lambda + x\lambda)\, R + (\lambda - 1).$$

Corollary 1.3 — Let $\beta = \frac{1}{2}$. Then we obtain a result of Swamy (1978, Theorem 5) which, when $\gamma = 1$ reduces to a result of Shah (1972, Theorem 5).

Corollary 1.4 — Let $\lambda = 0$. Then $f(z)$ is univalent and starlike for $|z| < R_t^{1/n}$, where R_t is the smallest positive root of the equation

$$xcR^3 + (nc + n + c + x - xc)\, R^2 + (nc + n + 1 - c - x)\, R - 1 = 0. \quad \ldots(7)$$

Sharpness of this Corollary can be proved by considering

$$f(z) = z(1 - z^n)/(1 + cz^n)\,(1 + z^n)^{(1-x)/n}, \; g(z) = z/(1 + z^n)^{(1-d)/n},$$

and $h(z) = z/(1 + z^n)^{(1-e)/n}$. Hence Theorem 1 is sharp at least for $\lambda = 0$.

For the special case $\gamma = n = 1$, Corollary 1.4 reduces to a result of Shaffer (1974, Theorem 4) which, when $\beta = 0$ reduces to Theorem 3 of Ratti (1968), and when $\beta = \frac{1}{2}$ reduces to Theorem 6 of Ratti (1968).

For the case $\gamma = \alpha = 1$, Corollary 1.4 reduces to Theorem 1 of Shaffer (1974).

Theorem 2 — Let $u(z) \in S^*(\alpha)$, $h(z) \in S^*(\gamma)$, and $zf(z)/[\lambda zf(z) + (1-\lambda)\, g(z)\, h(z)] \in P(\beta, n)$ and also $0 < c + \lambda \leqslant 1$, whenever $\lambda > 0$. *Hypothesis A* : Let $g(z)/u(z) \in P(\delta, n)$. *Hypothesis B* : Let $g(z)/u(z) \in H(\delta, n)$.

Then with either hypothesis $f(z)$ is univalent and starlike for $|z| < R_t^{1/n}$, where R_t is the smallest positive root of the equation

$$\begin{aligned} G(R) = {} & xcm(c+\lambda)\, R^5 \\ & + [(xc + xm + mc + mnc + nc)\,\lambda + xc^2 + mc^2 - xmc^2 + 2mnc^2 \\ & \qquad + mnc + nc^2]\, R^4 \\ & + [(x + c + m - xcm + 2mnc + 2nc + mn + n)\,\lambda + c^2 - xc^2 - mc^2 \\ & \qquad + 2mnc^2 + 2nc^2 - xm - mn + nc]\, R^3 \\ & + [(2n + 2mn + nc + mnc - xc - xm - mc + 1)\,\lambda + xm - x - m - c^2 \\ & \qquad - mnc - 2mn + nc^2 - 2n]\, R^2 \\ & + [(mn + n - c - x - m)\,\lambda + x + m - 1 - 2n - nc - mn]\, R + (1-\lambda) = 0 \end{aligned} \qquad \ldots(8)$$

where $x = d + e - 1$.

PROOF : Let $\phi(z) = g(z)/u(z)$. Therefore, we get

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqslant \operatorname{Re}\left\{\frac{zu'(z)}{u(z)}\right\} - \left|\frac{z\phi'(z)}{\phi(z)}\right|. \qquad \ldots(9)$$

For both hypotheses A and B we have by Lemma 1 and Lemma 3

$$\left|\frac{z\phi'(z)}{\phi(z)}\right| \leqslant \frac{(1+m)\, n\, |z|^n}{(1 + m|z|^n)(1 - |z|^n)}. \qquad \ldots(10)$$

Now proceeding as in the proof of Theorem 1 and by (9) we obtain

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \operatorname{Re}\left\{\frac{zu'(z)}{u(z)}\right\} + \operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} - \left|\frac{z\phi'(z)}{\phi(z)}\right| - \left|\frac{zp'(z)/p(z)}{1 - \lambda p(z)}\right| - 1. \qquad \ldots(11)$$

Using Lemma 2, Lemma 3, Lemma 4, and the estimate (10), we obtain

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{1 + xR}{1 + R} - \frac{(1+m)\, nR}{(1 + mR)(1 - R)} - \frac{(1+c)\, nR(1 - cR)}{(1 + cR)(1 - R)\,[(1 - cR) - \lambda(1 + R)]} \qquad \ldots(12)$$

provided $|z| < \min\left\{1, \left(\frac{1-\lambda}{c+\lambda}\right)^{1/n}\right\}$, where $R = |z|^n$, $x = d + e - 1$. Hence $\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ if $|z| < \min\left\{1, \left(\frac{1-\lambda}{c+\lambda}\right)^{1/n}\right\}$ and if right-hand side of (12) is greater than zero. The latter will hold if $G(R) > 0$. Since $G(0) = 1 - \lambda > 0$, $G(1) = 4n(1+m)(1+c)(c+\lambda-1) \leqslant 0$, and is easy to see that $G[(1-\lambda)/(c+\lambda)] < 0$ whenever $(1-\lambda) < (c+\lambda)$, it follows that $G(R) = 0$ has at least one root in $(0, M]$, where $M = \min\left\{1, \frac{1-\lambda}{c+\lambda}\right\}$. Let R_t be the smallest positive root of $G(R) = 0$, then $0 < R_t \leqslant M$ and $G(R) > 0$ for $0 \leqslant R < R_t$. That is $f(z)$ is univalent and starlike for $|z| < R_t^{1/n}$. This completes the proof of Theorem 2.

Corollary 2.1 — Let $\alpha = 1$, and $\gamma = 1$. Then we obtain Theorem 2 of Chen (1974).

Corollary 2.2 — Let $\delta = \beta = 0$. Then we obtain a result of the authors (Kulkarni and Swamy 1978, Theorem 1).

For the special case $\alpha = 1$, Corollary 2.2 reduces to a result of Swamy (1978, Theorem 1) which, when $\gamma = 1$ reduces to a result of Shah (1972, Theorem 1).

Corollary 2.3 — Let $\delta = 0$ and $\beta = \frac{1}{2}$. Then we obtain a result of the authors (Kulkarni and Swamy 1978, Theorem 3).

For the special case $\alpha = 1$, Corollary 2.3 reduces to a result of Swamy (1978, Theorem 4) which, when $\gamma = 1$ reduces to a result of Shah (1972, Theorem 4).

Corollary 2.4 — Let $\delta = \lambda = 0$ in Theorem 2. Then $f(z)$ is univalent and starlike for $|z| < R_t^{1/n}$, where R_t is the smallest positive root of the equation

$$xcR^3 + (x + c + n + 3nc - xc)R^2 + (nc + 3n + 1 - x - c)R - 1 = 0.$$

Sharpness of this Corollary can be proved by considering

$$f(z) = z(1 - z^n)^2/(1 + cz^n)(1 + z^n)^{(1+n-x)/n},$$

$$g(z) = z(1 - z^n)/(1 + z^n)^{(1+n-d)/n},$$

$$u(z) = z/(1 + z^n)^{(1-d)/n}, \text{ and } h(z) = z/(1 + z^n)^{(1-e)/n}.$$

For the special case $\gamma = n = 1$, Corollary 2.4, reduces to a result of Shaffer (1974, Theorem 5, Hypothesis B) which, when $\beta = 0$, reduces to Theorem 3.1 of Causey and Merkes (1970) and when $\beta = \frac{1}{2}$ reduces to Theorem 3.3 of Causey and Merkes (1970).

Corollary 2.5 — Let $\beta = \lambda = 0$ in Theorem 2. Then $f(z)$ is univalent and starlike for $| z | < R_t^{1/n}$, where R_t is the smallest positive root of the equation

$$xmR^3 + (x + m + n + 3mn - xm) R^2 + (mn + 3n + 1 - x - m) R - 1 = 0.$$

For the case $\gamma = n = 1$, Corollary 2.5 reduces to a result of Shaffer (1974, Theorem 5, Hypothesis A).

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