TORSIONAL VIBRATIONS OF A SEMI-INFINITE POROELASTIC CYLINDER

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Using Biot's theory, a general theory is developed for torsional vibrations of a semi-infinite poroelastic cylinder when a shear stress is suddenly applied to its end surface. Explicit expressions have been given for displacement, velocity and stresses.

1. INTRODUCTION

The study of torsional vibrations are of paramount importance both from theoretical and practical considerations. The least mode of torsional vibration is dispersionless and is used in delay lines for storing information in the form of pulses. It will receive a pulse from a transmitter and return the pulse to the receiving end, essentially undistorted but delayed by a time interval which is determined by the path length and velocity. The memory properties of delay lines have been used extensively in computers. Torsional vibrations are used for finding the imperfections of materials. When the materials are defective, they cause reflection or refraction. These are also used for measuring elastic constants of a material.

The dynamical equations of a poroelastic solid are given by Biot (1956). Recently, Thajuddin (1978) has developed a general theory for vibrations of a fluid-saturated poroelastic cylinder. This theory is then applied briefly to consider particular examples of extensional vibrations, flexural vibrations and screw vibrations. Taking Biot's theory, in the following, the problem of transient torsional vibrations of a poroelastic cylinder is studied using Laplace transform. Expressions for displacement component, particle velocity and stresses are given. On neglecting fluid effects, the results of classical theory due to Campbell and Tsao (1972) are obtained as a particular case.

2. SOLUTION OF THE PROBLEM

Let a cylindrical polar coordinate system (r, θ, z) be taken with z-axis along the axis of a semi-infinite poroelastic cylinder and the plane end given by $z = 0$. We consider the problem of torsional vibrations when a shear stress $\sigma_{\theta z}$ is suddenly applied to the plane base. The displacement fields of solid and liquid media are $(0, v(r, z, t), 0)$, $(0, V(r, z, t), 0)$ respectively. The only non-zero stress components are

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$$\left.\begin{aligned} &\text{(a)} \quad \sigma_{r\theta} = N\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \\ &\text{(b)} \quad \sigma_{\theta z} = N\frac{\partial v}{\partial z}. \end{aligned}\right\} \quad \ldots(1)$$

In this case, the equations of motion to be satisfied are (Biot 1956)

$$\left.\begin{aligned} N\nabla^2 v &= \frac{\partial^2}{\partial t^2}(\rho_{11}v + \rho_{12}V) + b\frac{\partial}{\partial t}(v - V) \\ 0 &= \frac{\partial^2}{\partial t^2}(\rho_{12}v + \rho_{22}V) - b\frac{\partial}{\partial t}(v - V) \end{aligned}\right\} \quad \ldots(2)$$

N is shear modulus, b the dissipative coefficient and ρ's are densities in Biot's theory.

Taking the Laplace transform of v, we have

$$\bar{v}(r, z, p) = \int_0^\infty v(r, z, t)\, e^{-pt}\, dt \quad \ldots(3)$$

and the inversion relation is

$$v(r, z, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{v}(r, z, p)\, e^{pt}\, dp. \quad \ldots(3a)$$

Similar transforms may be defined for other variables like V etc. Applying (3) to (2), one obtains

$$\frac{\partial^2\bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{v}}{\partial r} - \left(q_1^2 + \frac{1}{r^2}\right)\bar{v} + \frac{\partial^2\bar{v}}{\partial z^2} = 0 \quad \ldots(4)$$

where initially v, V, $\frac{\partial v}{\partial t}$, $\frac{\partial V}{\partial t}$ are taken to be zero, and

$$\left.\begin{aligned} q_1^2 &= \frac{p^2(\rho_3^2\, p + b\,\rho)}{N(\rho_{22}p + b)} \\ \rho_3^2 &= \rho_{11}\rho_{22} - \rho_{12}^2,\ \rho = \rho_{11} + 2\rho_{12} + \rho_{22} \end{aligned}\right\} \quad \ldots(5)$$

Let

$$\bar{v} = R_1(r)\, \bar{Z}_1(z, p) \quad \ldots(6)$$

Substituting (6) into (4) gives

$$\frac{1}{R_1}\left(\frac{d^2R_1}{dr^2} + \frac{1}{r}\frac{dR_1}{dr} - \frac{R_1}{r^2}\right) = -\frac{1}{\bar{Z}_1}\left(\frac{d^2\bar{Z}_1}{dz^2} - q_1^2\,\bar{Z}_1\right) = -K_n^2 \quad \ldots(7)$$

k_n is a constant. From (7), we have

$$\frac{d^2R_1}{dr^2} + \frac{1}{r}\frac{dR_1}{dr} + (k_n^2 - 1/r_1^2)\,R_1 = 0 \qquad \text{...(8)}$$

$$\frac{d^2\bar{Z}_1}{dz^2} - q_2^2\,\bar{Z}_1 = 0 \qquad \text{...(9)}$$

where

$$q_2^2 = k_n^2 + q_1^2\cdot \qquad \text{...(10)}$$

The solutions of (8) and (9) are

$$\left.\begin{aligned} R_1 &= R_0 = c_1 r \quad (k_n = 0) \\ R_1 &= c_2 J_1(k_n r) \quad (k_n \neq 0) \\ \bar{Z}_1 &= c_3\, e^{-q_2 z r} \end{aligned}\right\} \qquad \text{...(11)}$$

c_1, c_2, c_3 are arbitrary constants. The boundary conditions to be satisfied are

$$\left.\begin{aligned} &\text{(i)}\quad s = Qe + R\epsilon \\ &\text{(ii)}\quad \text{At } r = a,\ \sigma_{r\theta} = 0 \\ \text{and}\ &\text{(iii)}\quad \text{At } z = 0,\ \sigma_{\theta z} = f(r)\,H(t) \end{aligned}\right\} \qquad \text{...(12)}$$

where 'a' is radius of the cylinder and $H(t)$ is Heaviside unit-step function. Because the torsional vibrations are dilatationless, so $e = \epsilon = 0$ and hence excess pore-pressure $s = 0$. Equation (12 *ii*) gives the frequency equation to be

$$J_2(k_n a) = 0 \qquad \text{...(13)}$$

where J_2 is a Bessel function of first kind of order two. The values of k_n are given by

$$K_n = \frac{X_n}{a},\ n = 1, 2, 3 \ldots$$

where $\{X_n\}$ are the zeros of $J_2(x)$.

On making use of (6), (11), (1*b*) and (12*iii*), we obtain

$$c_3 = -\,1/pq_2. \qquad \text{...(14)}$$

Thus we have

$$\bar{v} = \begin{cases} \dfrac{-\,c_1 r\, e^{-q_1 z}}{pq_1}, & (k_n = 0) \\[2ex] -\,\dfrac{c_2 J_1(k_n r)\, e^{-q_2 z}}{pq_2}, & (k_n \neq 0) \end{cases} \qquad \text{...(15)}$$

Simplifying (10), we obtain

$$q_2^2 = a_1^2 \cdot \frac{(p - A)(p - B)(p - D)}{(p - E)} \qquad \text{...(16)}$$

where

$$a_1^2 = \frac{\rho_3^2}{N\rho_{22}}, \; E = -\frac{b}{\rho_{22}} \qquad \text{...(17)}$$

A, B, D are the real roots of

$$p^3 + \frac{b\rho}{\rho_3^2} p^2 + \frac{Nk_n^2 \rho_{22}}{\rho_3^2} p + \frac{Nk_n^2 b}{\rho_3^2} = 0 \qquad \text{...(18)}$$

The roots are determined from the numerical values of N, b, 'ρ' for different materials and mode of vibration k_n. Following Ewing *et al.* (§2.5, p. 44, 1957) for different combinations of A, B, D, E, we have the displacement component v after taking Laplace inversion, for $k_n \neq 0$, as

Case (i) — If $E < D < B < A$, then

$$v = c_2 J_1(k_n r)\left[\frac{1}{\pi a_1}\left\{\int_A^B \frac{\cos(a_1 A_{21} z)\, e^{xt}}{A_{21}}\, dx + \int_E^D \frac{\cos(a_1 A_{22} z)\, e^{xt}}{A_{22}}\, dx\right\} - \frac{e^{-k_n z}}{k_n}\right]. \qquad \text{...(19)}$$

Case (ii) — If $D < E < B < A$, then

$$v = c_2 J_1(k_n r)\left[\frac{1}{\pi a_1}\left\{\int_A^B \frac{\cos(a_1 A_{21} z)\, e^{xt}}{A_{21}}\, dx + \int_E^D \frac{\cos(a_1 A_{23} z)\, e^{xt}}{A_{23}}\, dx\right\} - \frac{e^{-k_n z}}{k_n}\right]. \qquad \text{...(20)}$$

Case (iii) — If $B < D < E < A$, then

$$v = c_2 J_1(k_n r)\left[\frac{1}{\pi a_1}\left\{\int_A^E \frac{\cos(a_1 A_{21} z)\, e^{xt}}{A_{21}}\, dx + \int_D^B \frac{\cos(a_1 A_{24} z)\, e^{xt}}{A_{24}}\, dx\right\} - \frac{e^{-k_n z}}{k_n}\right]. \qquad \text{...(21)}$$

Case (iv) — If $A < B < D < E$, then

$$v = c_2 J_1(k_n r)\left[\frac{1}{\pi a_1}\left\{\int_D^E \frac{\cos(a_1 A_{25} z)\, e^{xt}}{A_{25}}\,dx + \int_B^A \frac{\cos(a_1 A_{26} z)\, e^{xt}}{A_{26}}\,dx\right\} - \frac{e^{-k_n z}}{k_n}\right], \quad \ldots(22)$$

where

$$\left.\begin{aligned} &E_1 A_{21} = A_2 B_1 D_1,\ E_1 A_{22} = A_2 B_2 D_2,\ E_2 A_{23} = A_2 B_2 D_1, \\ &\qquad E_2 A_{24} = A_2 B_1 D_2,\ E_2 A_{25} = A_1 B_1 D_1,\ E_2 A_{26} = A_1 B_2 D_2 \\ &A_1 = \sqrt{x - A},\ B_1 = \sqrt{x - B},\ D_1 = \sqrt{x - D},\ E_1 = \sqrt{x - E} \\ &A_2 = \sqrt{A - x},\ B_2 = \sqrt{B - x},\ D_2 = \sqrt{D - x},\ E_2 = \sqrt{E - x}. \end{aligned}\right\} \quad \ldots(23)$$

For $k_n = 0$ proceeding on similar lines, we have

$$v = \frac{c_1 r}{a_1}\left[-\frac{1}{\pi}\int_E^F \frac{\cos(a_1 A_{27} z)\, e^{xt}}{x A_{27}}\,dx + a_1 z - \frac{E - F}{E^{1/2} F^{3/2}}\right] \quad \ldots(24)$$

where

$$E_2 A_{27} = x F_1,\ F_1 = \sqrt{x - F},\ E_2 = \sqrt{E - x},\ F = -\frac{b\rho}{\rho_3^2}. \quad \ldots(25)$$

E and a_1 are defined in (17). Of the two branch points E, F appearing in the integrand, it is easy to find F is less than E. The complete solution for v is then

$$v = \frac{A_0 r}{a_1}\left[-\frac{1}{\pi}\int_E^F \frac{\cos(a_1 A_{27} z)\, e^{xt}}{x A_{27}}\,dx + a_1 z + \frac{F - E}{E^{1/2} F^{3/2}}\right]$$

$$+ \sum_{n=1}^{\infty} A_n J_1(k_n r)\left[\frac{1}{\pi a_1}\left\{\int_A^B \frac{\cos(a_1 A_{21} z)\, e^{xt}}{A_{21}}\,dx + \int_E^D \frac{\cos(a_1 A_{22} z)\, e^{xt}}{A_{22}}\,dx\right\} - \frac{e^{-k_n z}}{k_n}\right] \quad \ldots(26)$$

for $E < D < B < A$.

where

$$A_0 = c_1,\ A_n = c_2$$

For other cases, it follows similarly. The particle velocity $v_1 \left(= \frac{\partial v}{\partial t}\right)$ is

$$v_1 = \frac{A_0 r}{a_1}\Bigg[-\frac{1}{\pi}\int_E^F \frac{\cos(a_1 A_{27} z)\, e^{\alpha t}}{A_{27}}\, dx + \sum_{n=1}^{\infty} A_n J_1(k_n r)$$

$$\times \frac{1}{\pi a_1}\left\{\int_A^B \frac{x\cos(a_1 A_{21} z)\, e^{\alpha t}}{A_{21}}\, dx + \int_E^D \frac{x \cos(a_1 A_{22} z)\, e^{\alpha t}}{A_{22}}\, dx\right\} - \frac{e^{-k_n z}}{k_n}\Bigg] \qquad \text{...(27)}$$

for $E < D < B < A$

For other combinations the particle velocity can be computed similarly. The stress components are $\sigma_{\theta z}\left(= N\dfrac{\partial v}{\partial z}\right)$

$$\sigma_{\theta z} = N\Bigg\langle A_0 r\left\{\frac{1}{\pi}\int_E^F \frac{\sin(a_1 A_{27} z)\, e^{\alpha t}}{x}\, dx + 1\right\} + \sum_{n=1}^{\infty} A_n J_1(k_n r)$$

$$\times\left[-\frac{1}{\pi}\left\{\int_A^B \sin(a_1 A_{21} z)\, e^{\alpha t}\, dx + \int_E^D \sin(a_1 A_{22} z)\, e^{\alpha t}\, dx\right\} + e^{-k_n z}\right]\Bigg\rangle \qquad \text{...(28)}$$

for $E < D < B < A$

and

$$\sigma_{r\theta} = -N\sum_{n=1}^{\infty} A_n k_n J_2(k_n r)\Bigg[\frac{1}{\pi a_1}\left\{\int_A^B \frac{\cos(a_1 A_{21} z)\, e^{\alpha t}}{A_{21}}\, dx\right.$$

$$\left. + \int_E^D \frac{\cos(a_1 A_{22} z)\, e^{\alpha t}}{A_{22}}\, dx\right\} - \frac{e^{-k_n z}}{k_n}\Bigg] \qquad \text{...(29)}$$

for $E < D < B < A$.

Similarly the stresses can be computed for other cases. The constants of integration A_0, A_n appearing in all the above are determined from of (12*iii*) by making use of orthogonality properties of Bessel functions. These are

$$A_0 = \frac{2}{N\pi a^4}\, T \qquad \text{...(30)}$$

where the applied torque T is

$$T = \int_0^a 2\pi r^2 f(r)\, dr \qquad \text{...(31)}$$

and

$$A_n = \frac{2}{Na^2 J_1^2\,(k_n a)} \int_0^a r f(r)\, J_1(k_n r)\, dr. \qquad \ldots(32)$$

Substituting for A_0 and A_n in (26), (27), (28), (29) one can compute the displacement, particle velocity and stresses very easily. This is the case dealt when the roots of the cubic eqn. (18) are real. There may be a possibility when two of them are complex conjugate and one is real which, of course, depends on the numerical values of N, b, 'ρ' for different types of materials and mode of vibration k_n. In the following, we consider this case. Let the roots of (18) be $A + iB$, $A - iB$, D. A and B are real numbers. Proceeding on the lines outlined above, we have

$$v = \frac{A_0 r}{a_1}\left[-\frac{1}{\pi}\int_E^F \frac{\cos(a_1 A_{27} z)\, e^{xt}}{x A_{27}}\, dx + a_1 z - \frac{E-F}{E^{1/2}\, F^{3/2}}\right] + \sum_{n=1}^{\infty} A_n J_1(k_n r)\, A_{29} \qquad \ldots(33)$$

where

$$A_{29} = \frac{1}{\pi a_1}\left[-\int_D^E \frac{\cos(a_1 A_{28} z)\, e^{xt}}{x A_{28}}\, dx + \int_0^{2B} \frac{e^M (G_1 \cos L + H_1 \sin L)}{G_4 \sqrt{G_3}\,\{A^2 + (B-x)^2\}}\, dx\right] - \frac{e^{-k_n z}}{k_n} \qquad \ldots(34)$$

$$\left.\begin{aligned}
&E_1 A_{28} = D_2 \sqrt{(x-A)^2 + B^2},\ M = At - \frac{a_1 z}{\sqrt{2}} \sqrt{G_3(G_4 + G_2)}\\
&L = (B - x)\, t + \frac{a_1 z}{\sqrt{2}} \sqrt{G_3(G_4 - G_2)}\\
&G_1 = (B - x)\, H_2 + A G_2,\ H_1 = (B - x)\, G_2 - A H_2\\
&G_3 = x(2B - x),\ G_4 = \sqrt{G_2^2 + H_2^2}\\
&G_2, H_2 = \frac{1}{\sqrt{2}}\, [H_3^2 + H_4^2)^{1/2} \pm H_3]^{1/2}\\
&H_3 = 1 + \frac{(E - D)(A - E)}{(B - x)^2 + (A - E)^2}\\
&H_4 = 1 + \frac{(E - D)(B - x)}{(B - x)^2 + (A - E^2)}.
\end{aligned}\right\} \qquad \ldots(35)$$

In this case, the particle velocity v_1 and stresses are given by

$$v_1 = \frac{A_0 r}{a_1}\Bigg[-\frac{1}{\pi}\int_E^F \frac{\cos(a_1 A_{27} z)\, e^{xt}}{A_{27}}\,dx$$
$$+\sum_{n=1}^{\infty} A_n J_1(k_n r)\Bigg[-\frac{1}{\pi a_1}\int_E^D \frac{x\cos(a_1 A_{26} z)\, e^{xt}}{A_{26}}\,dx$$
$$+\int_0^{2B} \frac{e^M\{(AH_1 - G_1(B-x))\sin L + (AG_1 + H_1(B-x))\cos L\}}{G_4\{A^2+(B-x)^2\}\sqrt{G_3}}\,dx \qquad \ldots(36)$$

$$\sigma_{\theta z} = N\Bigg\langle A_0 r\left\{\frac{1}{\pi}\int_E^F \frac{\sin(a_1 A_{27} z)\, e^{xt}}{x}\,dx + 1\right\}$$
$$+\sum_{n=1}^{\infty} A_n J_1(k_n r)\Bigg[\frac{1}{\pi a_1}\Bigg\{\int_D^E \frac{a_1\sin(a_1 A_{28} z)\, e^{xt}}{x}\,dx$$
$$+\int_0^{2B} \frac{\{-(G_1L_1+H_1M_1)\sin L+(H_1L_1-G_1M_1)\cos L\}e^M}{G_4\{A^2+(B-x)^2\}\sqrt{G_3}}\,dx\Bigg\} + e^{-k_n z}\Bigg]\Bigg\rangle \qquad \ldots(37)$$

$$\sigma_{r\theta} = -N\sum_{n=1}^{\infty} A_n k_n J_2(k_n r)\,.\,A_{29} \qquad \ldots(38)$$

where

$$L_1 = \frac{a_1}{\sqrt{2}}\{G_3(G_4 - G_2)\}^{1/2},\quad M_1 = -\frac{a_1}{\sqrt{2}}\{G_3(G_4 + G_2)\}^{1/2} \qquad \ldots(39)$$

A_{29} is given by (34).

On making use of boundary condition (12*iii*) and orthogonality properties of Bessel functions, we find

$$A_0 = \frac{2}{N\pi a^4}\,.\,T$$

and

$$A_n = \frac{2}{Na^2 J_1^2(k_n a)\,M_2^2}\int_0^a rf(r)\,J_1(k_n r)\,dr \qquad \ldots(40)$$

T is defined in (31). M_2^2 is given by

$$M_2^2 = 1 + \frac{1}{\pi a_1} \times \int_0^{2B} \frac{e^{At}\{-(G_1L_1 + H_1M_1)\sin((B-x)t) + (H_1L_1 - G_1H_1).\cos((B-x)t)\}}{G_4\{A^2 + (B-x)^2\}\sqrt{G_3}}\,dx \quad \ldots(41)$$

The value of A_0 in both the cases depend explicitly on the applied torque T whereas A_n does not. Following McCoy (1964), the contribution for the torque to be applied on the plane end by all the terms of the series is zero. However for $b = 0$, that is when dissipative nature of liquid medium is ignored, eqn. (18) will be a cubic equation in p no more and therefore a case of $b = 0$ has special interest. Putting $b = 0$ in (5), eqn. (15) has a similarity with Campbell and Tsao (1972). Proceeding on the lines of Campbell and Tsao (1972) the displacement component, velocity and stresses are given by

$$v = -A_0 r y_1 - \sum_{n=1}^{\infty} A_n J_1(k_n r) \int_0^{y_2} y(y^2 + z^2)^{-1/2} J_0(k_n y)\,dy \quad \ldots(42)$$

$$v_1 = -c\,[A_0 r + \sum_{n=1}^{\infty} A_n J_1(k_n r)\,J_0(k_n y_2)] \quad \ldots(43)$$

$$\sigma_{\theta z} = N\,[A_0 r + \sum_{n=1}^{\infty} A_n J_1(k_n r)\,\{1 - k_n z\,.\int_0^{y_2} (y^2 + z^2)^{-1/2} J_1(k_n y)\,dy\}] \quad \ldots(44)$$

$$\sigma_{r\theta} = N \sum_{n=1}^{\infty} A_n k_n J_2(k_n r) \int_0^{y_2} y(y^2 + z^2)^{-1/2} J_0(k_n y)\,dy \quad \ldots(45)$$

where

$$y_1 = ct - z,\ y_2 = (c^2t^2 - z^2)^{1/2},\ a_1^2\,c^2 = 1 \quad \ldots(46)$$

a_1 is given in (17). The constants A_0 and A_n are given by (30), (32). In all our calculations, neglecting fluid effects classical results follow atonce (Campbell and Tsao 1972).

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