TORSIONAL LOADING OF A SEMI-INFINITE POROELASTIC SOLID

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In this paper, the problem of torsional loading of a semi-infinite poroelastic solid is studied using integral transform technique. The classical problem is obtained as a particular case.

1. INTRODUCTION

The first study of torsional vibrations of an elastic half-space due to a periodic force was made by Reissner and Sagoci (1944) who considered the problem of periodic surface stress over a circular area and zero stress outside this area. This problem in terms of an integral equation has been studied by Sneddon (1947). Eason (1964) has solved the same problem for different kinds of surface stresses.

The study of torsional oscillations play an important role in several fields like Soil Mechanics, Mechanical Engineering etc. This is used for measuring elastic constants of a crystal. From the least mode of torsional vibrations, the shear modulus of a material can be found out.

The governing equations of a poroelastic solid are given by Biot (1956). A general theory is developed for vibrations of a fluid-saturated poroelastic cylinder by Thajuddin (1978). As a particular examples of the above, the problems of extensional vibrations, flexural vibrations and screw vibrations are studied. Using Biot's theory, in the following, torsional loading of a semi-infinite poroelastic solid subjected to impulsive force is solved using Laplace and Hankel transforms. The expression for displacement is given. On neglecting the fluid effects, the results of an elastic medium follow which are in agreement with Eason (1964).

2. SOLUTION OF THE PROBLEM

We use cylindrical polar co-ordinate system (r, θ, z). For torsional loading of a semi-infinite poroelastic solid, $z \geqslant 0$, the displacement fields of a solid and liquid are $(0, v(r, z, t), 0)$, $(0, V(r, z, t), 0)$ respectively. The only non-zero stress components in this case will be

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$$\left.\begin{aligned}\sigma_{r\theta} &= N\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\\ \sigma_{\theta z} &= N\frac{\partial v}{\partial z}\end{aligned}\right\} \quad \ldots(1)$$

N is the shear modulus of solid medium. Because the considered vibrations are shear waves the excess pore-pressure(s)

$$s = Qe + R\epsilon$$

developed in solid-liquid aggregate is zero. With this in view, the equations of motion of a poroelastic solid (1956) will now be

$$\left.\begin{aligned}N\,\nabla^2 v &= \frac{\partial^2}{\partial t^2}(\rho_{11}v + \rho_{12}V) + b\frac{\partial}{\partial t}(v - V)\\ 0 &= \frac{\partial^2}{\partial t^2}(\rho_{12}v + \rho_{22}V) - b\frac{\partial}{\partial t}(v - V)\end{aligned}\right\} \quad \ldots(2)$$

where b is dissipative coefficient and ρ's are density coefficients such that $\rho_{11} + \rho_{12}$ and $\rho_{12} + \rho_{22}$ represent densities of solid and liquid respectively. ρ_{12} is mass-coupling parameter.

Taking the Laplace transform followed by Hankel transform of v, V, $\sigma_{\theta z}$; we have

$$(v^{-1}, V^{-1}, \sigma_{\theta z}^{-1}) = \int_0^\infty \int_0^\infty (v, V, \sigma_{\theta z})\, rJ_1(\alpha r)\, e^{-pt}\, dr\, dt \quad \ldots(3)$$

The inverse transform of (3) is

$$(v, V, \sigma_{\theta z}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\infty (v^{-1}, V^{-1}, \sigma_{\theta z}^{-1})\, J_1(\alpha r)\, e^{pt}\, d\alpha\, dp \quad \ldots(4)$$

where c is positive constant chosen such that the singularities of the integrand lie to the left of line $p = c$ in the complex p-plane. Applying (3) to (2), we obtain

$$\left(\frac{d^2}{dz^2} - n^2\right) v^{-1} = 0 \quad \ldots(5)$$

where

$$\left.\begin{aligned}&\text{(i)} \quad n^2 = \alpha^2 + k^2\\ &\text{(ii)} \quad k^2 = \frac{p^2(\rho_3^2\, p + b\,\rho)}{N(\rho_{22}p + b)}\\ &\text{(iii)} \quad \rho_3^2 = \rho_{11}\rho_{22} - \rho_{12}^2, \;\; \rho = \rho_{11} + 2\rho_{12} + \rho_{22}.\end{aligned}\right\} \quad \ldots(6)$$

The bounded solution of (5) is

$$v^{-1} = c_1 \exp(-nz) \quad \ldots(7)$$

where c_1 is an arbitrary constant. The boundary conditions which determine the constant c_1 is chosen as

$$\sigma_{\theta z} = q(r, t) \text{ on } z = 0, \; t > 0 \qquad \text{...(8)}$$

where $q(r, t) = 0$ for $t < 0$. Applying (3) to second equation of (1), (8) and making use of (7), we have

$$c_1 = - q^{-1}/Nn. \qquad \text{...(9)}$$

The displacement from (7), (9) and (4) gives

$$v = - \frac{1}{2\pi iN} \int_{c-i\infty}^{c+i\infty} \int_0^\infty \frac{\alpha q^{-1} e^{-nz}}{n} J_1(\alpha r)\, e^{pt}\, d\alpha\, dp, \qquad \text{...(10)}$$

Similarly the stresses can be computed. Let us consider a special type of loading, that is impulsive loading. For this it is assumed that

$$q(r, t) = M(r)\, \delta(t) \qquad \text{...(11)}$$

where $\delta(t)$ is Dirac-delta function.

Applying (3) to (11), we have

$$q^{-1} = \bar{M} \qquad \text{...(12)}$$

where

$$\bar{M} = \int_0^\infty rM(r)\, J_1(\alpha r)\, dr. \qquad \text{...(13)}$$

Insertion of (12) into (10) gives

$$v = - \frac{1}{2\pi iN} \int_{c-i\infty}^{c+i\infty} \int_0^\infty \frac{\alpha \bar{M} e^{-nz}}{n} J_1(\alpha r)\, e^{pt}\, d\alpha\, dp. \qquad \text{...(14)}$$

Let

$$M(r) = \begin{cases} Tr, & r < a \\ 0, & r > a \end{cases} \qquad \text{...(15)}$$

T and a are constants. From (15) and (13), one obtains

$$\bar{M} = \frac{Ta^2 J_2(\alpha a)}{\alpha}. \qquad \text{...(16)}$$

Substituting (16) in (14) gives

$$v = - \frac{Ta^2}{2\pi iN} \int_{c-i\infty}^{c+i\infty} \int_0^\infty \frac{J_2(\alpha a)\, J_1(\alpha r)\, e^{-nz+pt}}{n}\, d\alpha\, dp \qquad \text{...(17)}$$

Using the relation (Watson 1958)

$$J_1(\alpha r)\, J_2(\alpha a) = \frac{1}{\pi} \int_0^{\pi} \frac{(a - r e^{-i\phi})\, e^{-i\phi}\, J_1(\alpha d)}{d}\, d\phi \qquad \ldots(18)$$

into (17), performing Laplace inversion and after necessary manipulations, we obtain (see Appendix)

$$v = \frac{Ta^2 \sqrt{\rho_{22}}}{\pi^2 \rho_3 \sqrt{N}} \int_0^{\pi} \int_A^B \frac{(a \cos\phi - r \cos 2\phi)(\cos\theta - \cos(f\theta))\, \sqrt{B - x}\, e^{\alpha t}}{d^2 x \sqrt{x - A}} .\, dx\, d\phi \qquad \ldots(19)$$

where

$$\left.\begin{aligned} & d^2 = a^2 + r^2 - 2ar \cos\phi,\ f = \sqrt{1 + \frac{d^2}{z^2}} \\ & A = -\frac{b\rho}{\rho_3^2},\ B = -\frac{b}{\rho_{22}},\ \theta = \frac{z \rho_3 x \sqrt{x - A}}{\sqrt{N}\, \rho_{22}(B - x)} . \end{aligned}\right\} \qquad \ldots(20)$$

Equation (19) gives displacement in terms of finite integrals. At the lower limit the integrand appears to have a division by zero but on taking the Taylor's series at $x = A$, it takes a finite value and hence the integral is a proper integral which, of course, can be computed numerically. Also the integrand turns out to be imaginary when $b = 0$, hence this case is to be studied independently. Proceeding as before, the displacement component in this case is seen to be

$$v = -\frac{1}{2\pi i N} \int_{c-i\infty}^{c+i\infty} \int_0^{\infty} \frac{\alpha q^{-1} e^{-n_1 z}}{n_1} J_1(\alpha r)\, e^{pt}\, d\alpha\, dp \qquad \ldots(21)$$

where

$$\left.\begin{aligned} & n_1^2 = \alpha^2 + k_1^2 \\ & k_1^2 = \frac{p^2}{k_2^2},\ k_2^2 = \frac{N \rho_{22}}{\rho_3^2} . \end{aligned}\right\} \qquad \ldots(22)$$

Equation (21) is in the same form as that of Eason (1964). Hence the displacement in the case of $\sigma_{\phi z} = Tr\delta(t)\, H(a - r)$, is

$$-\frac{Nv}{Tk_2 a} = \begin{cases} 0 & , r > y + a \\ 0 & , r < y - a \\ r/a, & r < a - y \end{cases}$$

(equation continued on p. 1151)

$$\left| \begin{array}{l} \dfrac{1}{2\pi ar}\Big\{(r^2 + a^2)(\pi - B_1) - | r^2 - a^2 | \pi \\ - 2ar \sin B_1 + 2 | r^2 - a^2 | \tan^{-1}\left[\left|\dfrac{r+a}{r-a}\right| \tan \dfrac{B_1}{2}\right] \\ - 2a^2 B_2\Big\}, | y - a | < r < y + a \end{array} \right. \qquad \ldots(23)$$

In the above $H(a - r)$ is Heaviside unit-step function and

$$\left. \begin{array}{l} \cos B_1 = \dfrac{r^2 + a^2 - y^2}{2ar}, \quad \cos B_2 = \dfrac{a^2 + y^2 - r^2}{2ay} \\ y = (k_2^2 t^2 - z^2)^{1/2}. \end{array} \right\} \qquad \ldots(24)$$

For $r > a - y$, $v = 0$ except for $| y - a | < r < y + a$. Substituting the value of y, it gives

$$R_1 < k_2 t < R_2$$

where

$$R_1 = \{(r - a)^2 + z^2\}^{1/2}, \quad R_2 = \{(r + a)^2 + z^2\}^{1/2}$$

Proceeding on similar lines, one can obtain the solutions for other types of loading considered by Eason (1964) without difficulty. Neglecting the fluid effects in our calculations, we obtain the classical result of Eason (1964).

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Appendix

Insertion of (18) into (17) and using a result given by Eason (1966),

$$v = -\frac{Ta^2}{2\pi^2 iN} \int_{c-i\infty}^{c+i\infty} e^{pt}\, dp$$

$$\times \int_0^{\pi} \frac{a \cos \phi - r \cos 2\phi}{d} I_{1/2}(B_1 k)\, K_{1/2}(B_2 k)\, d\phi \quad (z > 0) \qquad \text{...(A1)}$$

where

$$\left.\begin{aligned} 2B_1 &= (z^2 + d^2)^{1/2} - z, \; 2B_2 = (z^2 + d^2)^{1/2} + z \\ d^2 &= a^2 + r^2 - 2ar \cos \phi. \end{aligned}\right\} \qquad \text{...(A2)}$$

I, K are modified Bessel functions and a result from Gradshtyn *et al.* (1965, p. 719) is used. Interchanging the order of integration and using (Watson 1958, p. 80),

$$I_{1/2}(z) = \frac{1}{\sqrt{2\pi z}} (e^z - e^{-z}), \; K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} \qquad \text{...(A3)}$$

we have

$$v = -\frac{Ta^2}{4\pi^2 iN} \int_0^{\pi} \frac{a \cos \phi - r \cos 2\phi}{d \sqrt{B_1 B_2}}\, d\phi$$

$$\times \int_{c-i\infty}^{c+i\infty} \frac{e^{(B_1 - B_2)k} - e^{-(B_1 + B_2)k}}{k}\, e^{pt}\, dp. \qquad \text{...(A4)}$$

Substituting the value of k from second equation of (6),

$$v = -\frac{Ta^2}{4\pi^2 i} \sqrt{\frac{E_1}{ND_1}} \int_0^{\pi} \frac{a \cos \phi - r \cos 2\phi}{d \sqrt{B_1 B_2}}\, d\phi$$

$$\times \int_{c-i\infty}^{c+i\infty} \frac{\left[\exp\left(F_1 p \sqrt{\dfrac{p - A}{p - B}}\right) - \exp\left(- F_2 p \sqrt{\dfrac{p - A}{p - B}}\right)\right] e^{pt}}{p \sqrt{p - A}}\, dp \qquad \text{...(A5)}$$

where

$$\left.\begin{aligned} F_1 &= \frac{(B_1 - B_2) \sqrt{D_1}}{\sqrt{NE_1}}, \; F_2 = \frac{(B_1 + B_2) \sqrt{D_1}}{\sqrt{NE_1}} \\ A &= -\frac{b\rho}{\rho_3^2}, \; B = -\frac{b}{\rho_{22}}, \; D_1 = \rho_3^2, \; E_1 = \rho_{22} \end{aligned}\right\} \qquad \text{...(A6)}$$

B_1 and B_2 are given in (A2).

For Laplace inversion, we proceed on similar lines given in Ewing *et al.* (§2.5 p. 44, 1957). The poles are at $p = 0$, branch points are at $p = A, B$. Residue of the integrand, at $p = 0$, is zero. Of the two branch points A, B it is easy to find that $A < B$. The value of the integral is then

$$\int_A^B \frac{\left\{\cos\left(F_1 x \sqrt{\frac{x - A}{B - x}}\right) - \cos\left(F_2 x \sqrt{\frac{x - A}{B - x}}\right)\right\}}{ix\sqrt{x - A}} \sqrt{B - x}\, e^{xt}\, dx \quad \text{...(A7)}$$

Substituting (A7) into (A5) and using (A6), we have

$$v = \frac{Ta^2 \sqrt{\rho_{22}}}{\pi^2 \rho_3 \sqrt{N}}$$

$$\times \int_0^\pi \int_A^B \frac{(a \cos\phi - r\cos 2\phi)(\cos\theta - \cos(f\theta))\sqrt{B - x}}{d^2 x\sqrt{x - A}}\, dx\, d\phi \quad \text{...(A8)}$$

where f, θ are given in (20).