A NOTE ON THE SPACE-MATTER SPINOR*

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The purpose of this note is to present the spinor version of the space-matter tensor and its classification in terms of the principal null direction.

In this note we shall obtain the spinor equivalent of space-matter tensor and also give a classification of space-matter spinor in terms of the principal null directions.

1. THE SPACE-MATTER TENSOR

We assume that the metric

$$ds^2 = g_{ab}dx^a dx^b$$

of the space-time V_4 is reducible at a point to the Galilian form

$$ds^2 = -(dx_1)^2 - (dx_2)^2 - (dx_3)^2 + (dx_4)^2.$$

Let the field equations be

$$R_{ab} - \tfrac{1}{2} Rg_{ab} = \lambda T_{ab} \qquad \text{...(1.1)}$$

where λ is a constant and T_{ab} is the energy momentum tensor. On the contraction (1.1) yields

$$\lambda T = -R. \qquad \text{...(1.2)}$$

Introduce a fourth order tensor (Petrov 1969)

$$A_{abcd} = \frac{\lambda}{2}(g_{ac}T_{bd} + g_{bd}T_{ac} - g_{ad}T_{bc} - g_{bc}T_{ad}). \qquad \text{...(1.3)}$$

This tensor has the following properties :

$$A_{abcd} = -A_{bacd} = -A_{abdc} = A_{cdab} \qquad \text{...(1.4)}$$

$$A_{abcd} + A_{acbd} + A_{adbc} = 0. \qquad \text{...(1.5)}$$

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Contraction of (1.3) over b and d yields

$$A_{ac} = \lambda T_{ac} + \frac{\lambda}{2} T g_{ac} = \lambda T_{ac} - \frac{R}{2} g_{ac}. \qquad \text{...(1.6)}$$

Define a fourth order tensor (Petrov 1969)

$$P_{abcd} = R_{abcd} - A_{abcd} + \sigma(g_{ac}g_{bd} - g_{ad}g_{bc}). \qquad \text{...(1.7)}$$

This tensor is known as space-matter tensor. The first part of this tensor represents the curvature of the space and the second part represents the distribution and motion of the matter. This tensor has the following properties :

(i) $P_{abcd} = - P_{bacd} = - P_{abdc} = P_{cdab}$

$P_{abcd} + P_{acdb} + P_{adbc} = 0.$

(ii) $P_{ac} = R_{ac} - \lambda T_{ac} + \frac{R}{2} g_{ac} + 3\sigma g_{ac}$

$= (R + 3\sigma)\, g_{ac}.$

(iii) If the distribution and motion of the matter i.e. T_{ab} and the space-matter tensor P_{abcd} are given, then R_{abcd}, the curvature of the space is determined to within the scalar σ.

(iv) If $T_{ab} = 0$ and $\sigma = 0$, then P_{abcd} is the curvature of the empty space-time.

(v) If g_{ab} the metric tensor, σ the scalar and P_{abcd} are known then T_{ab} can be determined uniquely.

It has been shown by Ahsan (1977) that the space-matter tensor (1.7) can be decomposed as follows :

$$P_{abcd} = C_{abcd} + (g_{ad}R_{bc} + g_{bc}R_{ad} - g_{ac}R_{bd} - g_{bd}R_{ac}) + (\tfrac{2}{3}R + \sigma)(g_{ac}g_{bd} - g_{ad}g_{bc}). \qquad \text{...(1.8)}$$

2. THE SPACE-MATTER SPINOR

Following Penrose (1960), we have the following identification

$$g_{ab} \leftrightarrow \epsilon_{AB}\epsilon_{X'Y'}. \qquad \text{...(2.1)}$$

$$R_{abcd} \leftrightarrow R_{AW'BX'CY'DZ'} = \tfrac{1}{2}(\psi_{ABCD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \bar{\psi}_{W'X'Y'Z'}\epsilon_{AB}\epsilon_{CD} + \epsilon_{AB}\epsilon_{Y'Z'}\Phi_{CDW'X'} + \epsilon_{CD}\epsilon_{W'X'}\Phi_{ABY'Z'}) + \frac{R}{12}(\epsilon_{AC}\epsilon_{BD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{W'Z'}\epsilon_{X'Y'}) \qquad \text{...(2.2)}$$

where

$$\psi_{ABCD} = \psi_{(ABCD)} \qquad ...(2.3)$$

$$\Phi_{ABXY} = \Phi_{(AB)(XY')} = \bar{\Phi}_{ABX'Y'} \qquad ...(2.4)$$

$$R = R^{a}_{a} \cdot \qquad ...(2.5)$$

$$C_{abcd} \leftrightarrow C_{AW'BX'CY'DZ'} = \tfrac{1}{2}\left(\psi_{ABCD}\,\epsilon_{W'X'}\,\epsilon_{Y'Z'} + \bar{\psi}_{W'X'Y'Z'}\,\epsilon_{AB}\,\epsilon_{CD}\right). \quad ...(2.6)$$

$$R^{c}_{acb} = R_{ab} \leftrightarrow R_{AX'BY'} = \frac{R}{4}\,\epsilon_{AB}\,\epsilon_{X'Y'} - \Phi_{ABX'Y}. \qquad ...(2.7)$$

R_{abcd} and C_{abcd} are the Riemann and the Weyl curvature tensor respectively; ψ_{ABCD} and $\Phi_{ABX'Y'}$ are the Weyl and Einstein spinors, respectively.

Using (2.1 - 2.7), the space-matter tensor (1.7) corresponds to the following spinor expression

$$\begin{aligned} P_{abcd} \leftrightarrow P_{AW'BX'CY'DZ'} = {} & R_{AW'BX'CY'DZ'} + \tfrac{1}{2}\left(\epsilon_{AD}\chi_{BCX'Y'}\,\epsilon_{W'Z'}\right. \\ & \left. + \epsilon_{BC}\chi_{ADW'Z'}\,\epsilon_{X'Y'}\right) - \tfrac{1}{2}\left(\epsilon_{AC}\chi_{BDX'Z'}\,\epsilon_{WY'}\right. \\ & \left. + \epsilon_{BD}\chi_{ACWY'}\,\epsilon_{X'Z'}\right) \end{aligned} \qquad ...(2.8)$$

where

$$\chi_{BX'DZ'} = \Phi_{BX'DZ'} - \frac{\sigma}{2}\,\epsilon_{BD}\,\epsilon_{X'Z'}.$$

If, we choose $R_{ab} = 0 = \sigma$ in eqn. (1.8), then the space-matter spinor takes the form

$$P_{AW'BX'CY'DZ'} = \tfrac{1}{2}\left(\psi_{ABCD}\,\epsilon_{W'X'}\,\epsilon_{Y'Z'} + \bar{\psi}_{W'X'Y'Z'}\,\epsilon_{AB}\,\epsilon_{CD}\right) \qquad ...(2.9)$$

where ψ_{ABCD} is the Weyl spinor.

3. The Classification of the Space-matter Spinor

We shall now give a classification scheme for the space-matter spinor in terms of the principal null directions (Penrose 1960). From eqn. (2.9), it is clear that the classification problem in empty space-time for the space-matter spinor is equivalent to the classification of Weyl spinor. The classification of the space-matter spinor by means of principal null directions has been summarized in Table I.

In Table I, $\lambda \neq 0$ and 1-spinors represented by different kernel letters are not proportional to one another. The equation in the right hand column is satisfied if k^a is the real null vector corresponding to α^A (but in type I, to any of the four principal spinors, in type D, to either α^A or β^B).

TABLE I

Symbol for Petrov types	Form of ψ_{ABCD}	Equations satisfied by ψ_{ABCD}	Equations satisfied by P_{abcd}
I	$\psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$	$\psi_{ABCD}\alpha^B\alpha^C\alpha^D = \lambda\alpha_A$	$k_{[a}P_{b]cd[e}k_{f]}k^ck^d = 0$
II	$\psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}$	$\psi_{ABCD}\alpha^C\alpha^D = \lambda\alpha_A\alpha_B$	$P_{bcd[e}k_{f]}k^ck^d = 0$
D	$\psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\beta_{D)}$		
III	$\psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}$	$\psi_{ABCD}\alpha^D = \lambda\alpha_A\alpha_B\alpha_C$	$P_{bcd[e}k_{f]}k^d = 0$
N	$\psi_{ABCD} = \alpha_A\alpha_B\alpha_C\alpha_D$	$\psi_{ABCD} = \alpha_A\alpha_B\alpha_C\alpha_D$	$P_{bcde}k^e = 0.$

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