ON GENERALIZED POLYNOMIAL SET $D_n(x;\, a,\, k)$

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The object of the present paper is to furnish complementary argument theorem and the relation between polynomials of successive orders for the unified polynomials $D_n(x; a, k)$ of Karande and Thakare, whereby the corresponding results for Bernoulli, Euler, Carlitz's Eulerian polynomials and numbers are rendered intuitive.

1. INTRODUCTION

Karande and Thakare (1975) have stated the generalized polynomials $D_n(x; a, k)$ by the following relation

$$\sum_{n=0}^{\infty} D_n(x;\, a,\, k)\, \frac{t^n}{n!} = \frac{2(t/2)^k\, e^{xt}}{e^t - a} \qquad \ldots(1.1)$$

where a is a non-zero real number and k is an integer.

They have shown the following relationship between $D_n(x; a, k)$ and Bernoulli and other polynomials :

(i) Bernoulli polynomials : when $a = k = 1$, we have

$$D_n(x;\, a,\, k) = B_n(x). \qquad \ldots(1.2)$$

(ii) Bernoulli numbers : when $a = k = 1$, and $x = 0$, we have

$$D_n(0;\, 1,\, 1) = B_n.$$

(iii) Euler polynomials : when $a = -1$, $k = 0$, we have

$$D_n(x;\, a,\, k) = E_n(x) \qquad \ldots(1.3)$$

(iv) Genochhi numbers : when $-a = k = 1$, $x = 0$, we get

$$2D_n(0;\, -1,\, 1) = G_n$$

where these numbers are defined by

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n\, \frac{t^n}{n!}.$$

(v) Eulerian polynomials : when we put

$$k = 0, \quad a = \frac{1}{\xi} \text{ with } \xi \neq 1, \text{ we get}$$

$$D_n\left(x; \frac{1}{\xi}, 0\right) = \frac{2\xi}{\xi - 1} \phi_n(x, \xi)$$

where Eulerian polynomials $\{\phi_n(x, \xi)\}$ are defined by

$$\frac{1 - \xi}{1 - \xi e^t} e^{xt} = \sum_{n=0}^{\infty} \phi_n(x, \xi) \frac{t^n}{n!}$$

under suitable conditions (Carlitz 1962).

In the present paper an attempt has been made to prove the complementary argument theorem for the unified polynomial set $D_n(x; a, k)$. Moreover, the relations between polynomials of successive orders for the said unified polynomials have also been deduced.

2. Unified Polynomial Set $D_n(x; a, x)$

The generalized polynomial set $D_n(x; a, k)$ is defined by

$$\sum_{n=0}^{\infty} D_n(x; a, k) \frac{t^n}{n!} = \frac{2(t/2)^k \cdot e^{xt}}{e^t - a}. \qquad \ldots(2.1)$$

If in (2.1) we write $x + y$ for x, we obtain

$$\sum_{n=0}^{\infty} D_n(x + y; a, k) \frac{t^n}{n!} = e^{xt} \sum_{n=0}^{\infty} D_n(y; a, k) \frac{t^n}{n!}.$$

Equating the coefficients of t^n, we have

$$D_n(x + y; a, k) = D_n(y; a, k) + x \binom{n}{1} D_{n-1}(y; a, 1) + x^2 \binom{n}{2} D_{n-2}(y; a, 2) + \ldots + x^n \binom{n}{n} D_0(y; a, n). \qquad \ldots(2.2)$$

Putting $y = 0$, we obtain

$$D_n(x; a, k) = D_n(0; a, k) + x \binom{n}{1} D_{n-1}(0; a, 1) + x^2 \binom{n}{2} D_{n-2}(0, a, 2) + \ldots + x^n \binom{n}{n} D_0(0; a, n) \qquad \ldots(2.3)$$

which shows, unless $D_0(0; a, n) = 0$, that

$D_n(x; a, k)$ is actually of degree n.

Nörlund's operator $\underset{\omega}{\triangle}$:

It is defined by the relation

$$\underset{\omega}{\triangle} u(x) = \frac{u(x+\omega) - u(x)}{\omega}.$$

This symbol has the advantage that

$$\lim_{\omega \to 0} \underset{\omega}{\triangle} u(x) = Du(x), \ D \equiv \frac{d}{dx}.$$

The operator $\underset{\omega}{\nabla}$:

The definition of this operator is given by

$$\underset{\omega}{\nabla} u(x) = \tfrac{1}{2}\,[u(x) + u(x+\omega)].$$

If $\omega = 1$, we shall write $\triangle$ instead of $\underset{1}{\triangle}$ and ∇ instead of $\underset{1}{\nabla}$.

Operating on (2.1) with $\triangle$, we have

$$\sum_{n=0}^{\infty} \triangle D_n(x; a, k) \frac{t^n}{n!} = \frac{2(t/2)^k e^{xt}}{e^t - a} (e^t - 1). \qquad \text{...(2.4)}$$

Operating on (2.1) with ∇, we have

$$\sum_{n=0}^{\infty} \nabla D_n(x; a, k) \frac{t^n}{n!} = \frac{2(t/2)^k}{e^t - a} e^{xt} \left(\frac{e^t + 1}{2}\right). \qquad \text{...(2.5)}$$

3. The Complementary Argument Theorem

The arguments x and $\alpha - x$ are called complementary. We shall now prove that

$$D_n^{(\alpha)}(\alpha - x; a, k) = (-1)^n D_n^{(\alpha)}(x; a, k). \qquad \text{...(3.1)}$$

Proof : We can write (2.1) as

$$\sum_{n=0}^{\infty} D_n^{(\alpha)}(x; a, k) \frac{t^n}{n!} = \frac{\{2(t/2)^k\}^\alpha e^{\alpha t}}{(e^t - a)^\alpha}. \qquad \text{...(3.2)}$$

Now $$\sum_{n=0}^{\infty} D_n^{(\alpha)}(\alpha - x; a, k)\frac{t^n}{n!} = \frac{\{2(t/2)^k\}^{\alpha} e^{(\alpha-x)t}}{(e^t - a)^{\alpha}}$$

$$= \frac{\{2(t/2)^k\}^{\alpha} e^{-xt}}{(1 - ae^{-t})^{\alpha}}$$

$$= \frac{\{-(2(t/2)^k\}^{\alpha} e^{-xt}}{(ae^{-t} - 1)^{\alpha}}$$

$$= \sum_{n=0}^{\infty} D_n^{(\alpha)}(x; a, k)\frac{(-t)^n}{n!}.$$

Equating coefficients of t^n, we have the required result. If in (3.1), we put $x = 0$, $n = 2\mu$, we get

$$D_{2\mu}^{(\alpha)}(\alpha; a, k) = D_{2\mu}^{(\alpha)}(0; a, k). \qquad \ldots(3.3)$$

Thus $D_{2\mu}^{(\alpha)}(x; a, k) - D_{2\mu}^{(\alpha)}(0; a, k)$ has zeros at $x = \alpha$, $x = 0$.

Again with $x = \frac{\alpha}{2}$, $n = 2\mu + 1$, we have

$$D_{2\mu+1}^{(\alpha)}\left(\frac{\alpha}{2}; a, k\right) = - D_{2\mu+1}^{(\alpha)}\left(\frac{\alpha}{2}; a, k\right).$$

Thus $$D_{2\mu+1}^{(\alpha)}\left(\frac{\alpha}{2}; a, k\right) = 0. \qquad \ldots(3.4)$$

4. The Relation between Polynomials of Successive Orders

Differentiating both the sides of (3.2) with respect to t and multiplying by t, we have

$$\sum_{n=1}^{\infty} D_n^{(\alpha)}(x; a, k)\frac{t^n}{(n-1)!}$$

$$= \alpha k\left[\frac{e^{xt}\{2(t/2)^k\}^{\alpha}}{(e^t - a)^{\alpha}}\right] + xt\left[\frac{e^{xt}\{2(t/2)^k\}^{\alpha}}{(e^t - a)^{\alpha}}\right]$$

$$- \alpha t \cdot \frac{1}{\{(2(t/2)^k\}}\left[\frac{e^{(x+1)t}\{2(t/2)^k\}^{\alpha+1}}{(e^t - a)^{\alpha+1}}\right]$$

$$= \alpha k\sum_{n=0}^{\infty} D_n^{(\alpha)}(x; a, k)\frac{t^n}{n!} + xt\sum_{n=0}^{\infty} D_n^{(\alpha)}(x; a, k)\frac{t^n}{n!}$$

$$- \alpha t \cdot \frac{1}{\{(t/2)^k\}}\sum_{n=0}^{\infty} D_n^{(\alpha+1)}(x + 1; a, k)\frac{t^n}{n!}.$$

Equating coefficients of t^n,

$$nD_n^{(\alpha)}(x; a, k) = \alpha k D_n^{(\alpha)}(x; a, k) + xn D_{n-1}^{(\alpha)}(x; a, k) - \alpha t \cdot \frac{1}{\{2(t/2)^k\}} D_n^{(\alpha+1)}(x+1; a, k). \quad \ldots(4.1)$$

We know that

$$D_n^{(\alpha+1)}(x+1; a, k) = D_n^{(\alpha+1)}(x; a, k) + n D_{n-1}^{(\alpha)}(x; a, k).$$

Thus we have

$$\frac{t}{\{2(t/2)^k\}} D_n^{(\alpha+1)}(x; a, k) = \left(k - \frac{n}{\alpha}\right) D_n^{(\alpha)}(x; a, k) + n\left(\frac{x}{\alpha} - \frac{t}{\{2(t/2)^k\}}\right) D_{n-1}^{(\alpha)}(x; a, k) \quad \ldots(4.2)$$

which is the required relation between polynomials of orders α and $\alpha + 1$.

Putting $x = 0$, we have

$$\frac{t}{\{2(t/2)^k\}} D_n^{(\alpha+1)}(0, a, k) = \left(k - \frac{n}{\alpha}\right) D_n^{(\alpha)}(0; a, k) - \frac{nt}{\{2(t/2)^k\}} D_{n-1}^{(\alpha)}(0; a, k). \quad \ldots(4.3)$$

Again from (4.1), putting $x = 0$, we obtain

$$\frac{t}{\{2(t/2)^k\}} D_n^{(\alpha+1)}(1; a, k) = \left(k - \frac{n}{\alpha}\right) D_n^{(\alpha)}(0; a, k)$$

or, writing $\alpha + n$ for α,

$$\frac{t}{\{2(t/2)^k\}} D_n^{(\alpha+n+1)}(1; a, k) = \left(k - \frac{n}{\alpha + n}\right) D_n^{\alpha+n}(0; a, k). \quad \ldots(4.4)$$

References

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