ON THE STABILITY OF GENERALIZED HYDROMAGNETIC THERMAL SHEAR FLOWS

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The paper presents a mathematical analysis of the stability of generalized hydromagnetic thermal shear flows. The physical configuration is that of a horizontal layer of an incompressible inviscid heat conducting fluid of zero electrical resistivity in which there is a differential streaming $U(z)$ in the horizontal direction and density variation $\rho_0 f(z)$ in the vertical direction while the entire system is confined between two horizontal boundaries of different but uniform temperatures with the temperature of the lower boundary greater than that of the upper one in the presence of a uniform horizontal magnetic field, ρ_0 being a positive constant having the dimensions of density and $U(z)$ and $f(z)$ being continuous functions of the vertical coordinate z with $df/dz < 0$ everywhere in the flow domain. Results concerning the validity of the principle of exchange of stabilities, existence of non-oscillatory modes and the bounds for the phase velocity of disturbances, etc. are mathematically established for certain classes of $U(z)$.

Introduction

The stability of parallel shear flow of an inviscid heterogeneous incompressible fluid with stable density stratification to infinitesimal non-divergent disturbances has pervaded the scientific literature in the recent past on account of its importance in the fields of meteorology and oceanography. The fundamental works of Taylor (1931), Goldstein (1931), Drazin (1958), Miles (1961), Howard (1961) and others have clearly demonstrated that what decides the stability of this configuration is the numerical value of the non-dimensional parameter $J = g\beta \Big/ \left(\frac{dU}{dz}\right)^2$, called the Richardson number where g is the acceleration due to gravity, z is the vertical coordinate, ρ is the density and $\beta = -\frac{1}{\rho}\frac{d\rho}{dz}$ which is positive everywhere in the flow domain. For $J > \frac{1}{4}$ everywhere in the region of flow the system is stable while it may be unstable otherwise. For an excellent review of the previous work on the subject one is referred to Yih (1969). Gupta *et al.* (1977) examined the problem by taking into account changes in density due to thermal effects of moderate amounts

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(say 10°). This work is based on the rather plausible hypothesis that in reality a fluid is necessarily heat conducting and with a non-zero coefficient of thermal volume expansion and one may not always be justified in neglecting the small changes in density that are accounted for by the perturbation in the initial incompressible heterogeneity which themselves are small but still relevantly retained in the governing equations because its gravitational effects need not be small. Sufficient conditions have been obtained for the nonvalidity of the principle of exchange of stabilities. A marginal state solution of the governing equations under the above conditions is given and interpreted.

In the present paper we examine the problem treated by Gupta-Banerjee and others in the presence of a uniform horizontal magnetic field. The main results established are as follows :

(a) The principle of exchange of stabilities is not satisfied if U and $D^2U\left(D \equiv \frac{d}{dz}\right)$ do not change sign anywhere [0, 1] and $UD^2U < 0$ everywhere in [0, 1].

(b) If U is linear and positive (or negative) throughout the flow domain or satisfies conditions in result (a) above then nonoscillatory modes (if exist) are unstable.

(c) For linear velocity profiles the phase velocity c_r of an arbitrary neutral wave must lie within the minimum and maximum values of the streaming velocity in the flow domain. Further the phase velocity of an arbitrary neutral wave must satisfy $c_r > U_{min}$ or $c_r < U_{max}$ according as $U > 0$ and $D^2U < 0$ or $U < 0$ and $D^2U > 0$ everywhere in [0, 1]. The latter results also hold good for an arbitrary stable wave according as $D^2U < 0$ or $D^2U > 0$ everywhere in [0, 1].

(d) For small values of the coefficient of thermometric conductivity the phase velocity of an arbitrary stable wave must lie within the minimum and maximum values of the streaming velocity, the velocity being any general one.

(e) The stability of generalized thermal shear flows is not influenced by the presence of magnetic field in the transverse direction for disturbances with $k = k_x$, k being the wavenumber.

The Physical Problem and Its Formulation

An infinite horizontal layer of depth d of an initially stratified inviscid incompressible heat conducting fluid of zero electrical resistivity with a differential streaming $U(z)$ in the horizontal direction and density variation $\rho_0 f(z)$ in the vertical direction is confined between two horizontal boundaries of different but uniform temperatures with the temperature of the lower boundary greater than that of the upper one in the presence of a uniform horizontal magnetic field, ρ_0 being a positive constant having the dimensions of density and $f(z)$ is a monotonically decreasing function of the vertical coordinate z with the normalizing condition $f(0) = 1$ without any loss of generality. The problem is to investigate the stability of this initial stationary state. Throughout our mathematical analysis we shall assume $f(z)$ and $U(z)$ are respectively once and twice continuously differentiable everywhere in the flow domain.

Let the origin be taken on the lower boundary $z = 0$ with the positive direction of the z-axis along the vertically upward direction. Let $z = d(> 0)$ denote the upper boundary and T_0 and $T_1(< T_0)$ respectively denote the uniform temperatures of the lower and upper boundaries. The xy plane then constitutes the horizontal plane $z = 0$. Now for the problems of the type that we are considering it will suffice to work with the relevant equations in the Boussinesq approximation (Chandrasekhar 1961). Thus the relevant equations of momentum, incompressibility, heat conduction, state (Banerjee, 1971, 1972, 1973), magnetic induction and continuity which govern the problem are given by

$$\rho_0 \left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \rho X_i + \frac{\mu_e}{4\pi} H_j \frac{\partial H_i}{\partial x_j} - \frac{\mu_e}{8\pi} \frac{\partial}{\partial x_i}(H_j H_j) \quad \ldots(1)$$

$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = 0 \quad \ldots(2)$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \frac{K_1}{\rho_0 c_v} \nabla^2 T \quad \ldots(3)$$

$$\rho = \rho_0 [f(z) + \alpha(T_0 - T)] \quad \ldots(4)$$

$$\frac{\partial H_i}{\partial t} + u_j \frac{\partial H_j}{\partial x_j} = H_j \frac{\partial u_i}{\partial x_j} \quad \ldots(5)$$

$$\frac{\partial H_j}{\partial x_j} = 0 \quad \ldots(6)$$

$$\frac{\partial u_j}{\partial x_j} = 0 \quad \ldots(7)$$

where x_j ($j = 1, 2, 3$) respectively denote the x, y and z coordinates,

$$u_j,\ X_i(= (0, 0, -g)),\ H_i(i = 1, 2, 3;\ j = 1, 2, 3)$$

respectively denote the components of the velocity, external force and magnetic field in the x, y and z directions, T the temperature, ρ the density, p the pressure, μ_e the magnetic permeability and α and K_1 respectively stand for the coefficients of volume expansion and thermal conductivity and c_v the specific heat at constant volume.

Clearly the initial stationary state whose stability we wish to examine is then characterized by the following solutions for the velocity, temperature, density, pressure and magnetic fields respectively :

$$\left.\begin{aligned} u_i &= (u, v, w) \equiv (U, 0, 0) \\ T &= T_0 - \beta z, \\ \rho &= \rho_0 [f(z) + \alpha\beta z] \\ p &= -\int g\rho \, dz \\ H_i &= (H, 0, 0) \end{aligned}\right\} \quad \ldots(8)$$

where $\beta = \frac{T_0 - T_1}{d} > 0$ is the maintained uniform adverse temperature gradient.

Let the initial state described by eqns. (8) be slightly perturbed so that the perturbed state denoted by primed symbols is given by

$$\left.\begin{aligned} u'_j &= (U + u, v, w) \\ T' &= T + \theta \\ \rho' &= \rho_0\left[f(z) + \frac{\delta\rho}{\rho_0} + \alpha(T_0 - T - \theta)\right] \\ p' &= p + \delta p \\ H' &= (H + h_x, h_y, h_z) \end{aligned}\right\} \quad \ldots(9)$$

where (u, v, w), θ, $\delta\rho$, δp and (h_x, h_y, h_z) are the respective perturbations in velocity, temperature, initial density, pressure and magnetic fields.

The linearized perturbation equations of momentum, incompressibility, heat conduction, magnetic induction and continuity are then given by

$$\rho_0 \left[\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + w \frac{dU}{dz}\right] = \frac{\mu_e H}{4\pi} \frac{\partial h_x}{\partial x} - \frac{\partial \varpi}{\partial x} \quad \ldots(10)$$

$$\rho_0 \left[\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x}\right] = \frac{\mu_e H}{4\pi} \frac{\partial h_y}{\partial x} - \frac{\partial \varpi}{\partial y} \quad \ldots(11)$$

$$\rho_0 \left[\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x}\right] = \frac{\mu_e H}{4\pi} \frac{\partial h_z}{\partial x} - \frac{\partial \varpi}{\partial z} - g\delta\rho + g\alpha\rho_0\theta \quad \ldots(12)$$

$$\frac{\partial}{\partial t}(\delta\rho) + U \frac{\partial}{\partial x}(\delta\rho) + \rho_0 w \frac{df}{dz} = 0 \quad \ldots(13)$$

$$\frac{\partial \theta}{\partial t} - \beta w + U \frac{\partial \theta}{\partial x} = \kappa \nabla^2 \theta \quad \ldots(14)$$

$$\frac{\partial h_x}{\partial t} + U \frac{\partial h_x}{\partial x} = h_z \frac{dU}{dz} + H \frac{\partial u}{\partial x} \quad \ldots(15)$$

$$\frac{\partial h_y}{\partial t} + U \frac{\partial h_y}{\partial x} = H \frac{\partial v}{\partial x} \quad \ldots(16)$$

$$\frac{\partial h_z}{\partial t} + U\frac{\partial h_z}{\partial x} = H\frac{\partial w}{\partial x} \qquad \text{...(17)}$$

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} = 0 \qquad \text{...(18)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{...(19)}$$

where $\varpi = \delta p + \frac{\mu_e H}{4\pi} h_x$ and $\kappa = \frac{K_1}{\rho_0 c_v}$.

Analysing the perturbations in terms of normal modes by seeking solutions whose dependence on x, y and t is given by

$$\exp\,[i(k_x x + k_y y + nt)] \qquad \text{...(20)}$$

eqns. (10) – (19) become

$$i\rho_0 u\,[n + k_x U] + \rho_0 w\frac{dU}{dz} = ik_x\frac{\mu_e H}{4\pi}h_x - ik_x\varpi \qquad \text{...(21)}$$

$$i\rho_0 v\,[n + k_x U] = ik_x\frac{\mu_e H}{4\pi}h_y - ik_y\varpi \qquad \text{...(22)}$$

$$i\rho_0 w\,[n + k_x U] = -\frac{d\varpi}{dz} - g\delta\rho + g\alpha\rho_0\theta + ik_x\frac{\mu_e H}{4\pi}h_z \qquad \text{...(23)}$$

$$i\delta\rho\,[n + k_x U] = -\rho_0 w\frac{df}{dz} \qquad \text{...(24)}$$

$$i\,[n + k_x U]\,\theta - \beta w = \kappa\left(\frac{d^2}{dz^2} - k^2\right)\theta \qquad \text{...(25)}$$

$$i\,[n + k_x U]\,h_x = h_z\frac{dU}{dz} + ik_x H u \qquad \text{...(26)}$$

$$i\,[n + k_x U]\,h_y = ik_x H v \qquad \text{...(27)}$$

$$i\,[n + k_x U]\,h_z = ik_x H w \qquad \text{...(28)}$$

$$i\,[k_x h_x + k_y h_y] = -\frac{dh_z}{dz} \qquad \text{...(29)}$$

$$i\,[k_x u + k_y v] = -\frac{dw}{dz} \qquad \text{...(30)}$$

where $k = \sqrt{k_x^2 + k_y^2}$ is the wavenumber of the perturbation, n is a constant which can be complex and u, v, w, δp, h_x, h_y, h_z, θ and $\delta\rho$ are now functions of z only. Further we shall take into account the disturbances for which $k = k_x$.

Using the nondimensional quantities defined by

$$\left.\begin{aligned} &U_* = \frac{Ud}{\kappa};\ \sigma_* = \frac{ind^2}{\kappa};\ kd = a_*;\ D_* = d\,\frac{d}{dz};\\ &R_1 = \frac{g\alpha\beta d^4}{\kappa^2};\ R_2 = \frac{gd^4\,\dfrac{df}{dz}}{\kappa^2};\ Q = \frac{\mu_e H^2 d^2}{4\pi\rho_0\kappa^2};\ w_* = \frac{\beta d^2}{\kappa}\,w;\\ \text{and}\quad &\theta_* = \theta; \end{aligned}\right\} \qquad ...(31)$$

and dropping the asterisk for convenience in writing, the system of eqns. (21) - (30) yield the following equations :

$$(U - c)(D^2 - a^2)\,w - wD^2U - Q(D^2 - a^2)\left(\frac{w}{U - c}\right) - iR_1 a\theta - R_2\,\frac{w}{U - c} = 0 \qquad ...(32)$$

$$[D^2 - a^2 - ia(U - c)]\,\theta = -\,w \qquad ...(33)$$

where $\quad c = \dfrac{i\sigma}{a}.$

Solution of eqns. (32) and (33) must be sought subject to the following boundary conditions

$$w = 0 = \theta \ \text{ at } z = 0 \text{ and } z = 1. \qquad ...(34)$$

Equations (32) and (33) together with boundary conditions (34) present an eigenvalue problem for c for given values of the other parameters and a given state of the system is stable, marginal or unstable provided that the imaginary part c_i of c is negative, zero or positive respectively. Further if $c_i = 0$ implies the real part c_r of c also equal to zero for every wave number a, then the principle of exchange of stabilities is valid, otherwise, we will have overstability at least when instability sets in as certain modes.

We now prove the following theorems

Theorem 1 — The principle of exchange of stabilities is not satisfied if U and D^2U do not change sign anywhere in [0, 1] and $UD^2U < 0$ everywhere in [0, 1].

PROOF : If possible let the principle of exchange of stabilities be satisfied so that $c = 0$ is allowed by the governing equations and boundary conditions.

We then have from eqns. (32) and (33)

$$U(D^2 - a^2)\,w - wD^2U - Q(D^2 - a^2)\left(\frac{w}{U}\right) - iR_1 a\theta - R_2\,\frac{w}{U} = 0 \qquad ...(35)$$

$$[D^2 - a^2 - iaU]\,\theta = -\,w. \qquad ...(36)$$

Using the transformations

$$\left.\begin{aligned} w &= UF \\ \text{and} \qquad \theta &= U\Theta \end{aligned}\right\} \qquad \ldots(37)$$

eqns. (35) and (36) become

$$D\,[U^2DF] - a^2U^2F - R_2F - Q(D^2 - a^2)\,F = iR_1aU\Theta \qquad \ldots(38)$$

$$[D^2 - a^2 - iaU]\,(U\Theta) = -\,UF \qquad \ldots(39)$$

with $F = 0 = \Theta$ at $z = 0$ and $z = 1$. ...(40)

Multiplying eqn. (38) by F^* (the complex conjugate of F) and integrating over the vertical range of z we get

$$\int_0^1 F^*D\,[U^2DF]\,dz - a^2\int_0^1 U^2\,|\,F\,|^2\,dz - \int_0^1 R_2\,|\,F\,|^2\,dz - Q\int_0^1 F^*(D^2 - a^2)\,F\,dz = iR_1a\int_0^1 F^*U\Theta\,dz. \qquad \ldots(41)$$

From eqn. (39) we have

$$\int_0^1 U\Theta\,F^*dz = -\int_0^1 \Theta\,[D^2 - a^2 + iaU]\,(U\Theta^*)\,dz. \qquad \ldots(42)$$

Substituting from eqn. (42) into eqn. (41) we get

$$\int_0^1 F^*D\,[U^2DF]\,dz - a^2\int_0^1 U^2\,|\,F\,|^2\,dz - \int_0^1 R_2\,|\,F\,|^2\,dz - Q\int_0^1 F^*(D^2 - a^2)\,F\,dz = -\,iR_1a\int_0^1 \Theta\,[D^2 - a^2 + iaU]\,(U\Theta^*)\,dz. \qquad \ldots(43)$$

Integrating by parts and using the boundary conditions (40) we have

$$\int_0^1 F^*D\,[U^2DF]\,dz = -\int_0^1 U^2\,|\,DF\,|^2\,dz \qquad \ldots(44)$$

$$\int_0^1 F^*\,[D^2 - a^2]\,F\,dz = -\int_0^1 [\,|\,DF\,|^2 + a^2\,|\,F\,|^2]\,dz \qquad \ldots(45)$$

$$\int_0^1 \Theta\,D^2\,[U\Theta^*]\,dz = -\int_0^1 U\,|\,D\Theta\,|^2\,dz - I$$

where $I = \int_0^1 \Theta^*DUD\Theta\,dz = -\int_0^1 \Theta\,D[\Theta^*DU]\,dz = -\int_0^1 (D^2U)\,|\,\Theta\,|^2\,dz - I^*$

so that real part $I = -\frac{1}{2}\int_0^1 (D^2U)\mid \text{Ⓗ} \mid^2 dz$

and hence

$$\text{Real part} \quad [\int_0^1 \text{Ⓗ}\, D^2[U\text{Ⓗ}^*]\, dz] = -\int_0^1 U \mid D\text{Ⓗ} \mid^2 dz + \frac{1}{2}\int_0^1 (D^2U)\mid \text{Ⓗ} \mid^2 dz. \quad ...(46)$$

Substituting from eqns. (44), (45) and (46) in eqn. (43) and equating the imaginary parts of the resulting equation we get

$$\int_0^1 U \mid D\text{Ⓗ} \mid^2 dz - \frac{1}{2}\int_0^1 (D^2U)\mid \text{Ⓗ} \mid^2 dz + a^2\int_0^1 U \mid \text{Ⓗ} \mid^2 dz = 0. \quad ...(47)$$

Equation (47) clearly cannot hold under the conditions of the theorem. Thus our starting assumption, viz. the principle of exchange of stabilities is satisfied is wrong.

This proves the theorem.

Remarks : (1) From integral relation (47) we also deduce that the principle of exchange of stabilities is not satisfied if $U > 0$ (or < 0) and $D^2U = 0$ everywhere in [0, 1].

(2) Theorem 1 and Remark 1 are independent of the signs of R_1 and R_2.

Theorem 2 — If (i) U satisfies conditions of Theorem 1 or Remark 1; (ii) $c_r = 0$ (i.e. non-oscillatory modes exist); then $c_i > 0$. In other words under the conditions of the theorem the system is unstable.

PROOF : Under the conditions of the theorem

$$c = ic_i \neq 0$$

so that eqns. (32) and (33) now become

$$(U - ic_i)(D^2 - a^2)w - wD^2U - Q(D^2 - a^2)\left(\frac{w}{U - ic_i}\right) - iR_1a\theta$$
$$- R_2\frac{w}{U - ic_i} = 0 \quad ...(48)$$

and

$$[D^2 - a^2 - ia(U - ic_i)]\,\theta = -w. \quad ...(49)$$

Using the transformations

$$w = (U - ic_i)\,F$$

and

$$\theta = (U + ic_i)\,\text{Ⓗ}$$

eqns. (48) and (49) become

$$D[(U - ic_i)^2 DF] - a^2(U - ic_i)^2 F - R_2F - Q(D^2 - a^2)F$$
$$- iR_1a(U + ic_i)\,\text{Ⓗ} = 0 \quad ...(50)$$

and $$[D^2 - a^2 - ia(U - ic_i)]\,(U + ic_i)\,\Theta = -\,(U - ic_i)\,F. \qquad \ldots(51)$$

Proceeding exactly as in Theorem 1 it follows that

$$-\int_0^1 [U^2 - c_i^2 - 2ic_iU]\,[\,|DF|^2 + a^2|F|^2]\,dz - \int_0^1 R_2\,|F|^2\,dz$$
$$+\,Q\int_0^1 [\,|DF|^2 + a^2|F|^2]\,dz + iR_1a\left\{\int_0^1 \Theta\, D^2[(U - ic_i)\,\Theta^*]\,dz\right.$$
$$\left. -\,a^2\int_0^1 (U - ic_i)\,|\Theta|^2\,dz + ia\int_0^1 (U^2 + c_i^2)\,|\Theta|^2\,dz\right\} = 0. \qquad \ldots(52)$$

Further

Real part $$\left[\int_0^1 \Theta\, D^2[(U - ic_i)\,\Theta^*]\,dz = -\int_0^1 U\,|D\Theta|^2\,dz + \tfrac{1}{2}\int_0^1 (D^2U)\,|\Theta|^2\,dz. \qquad \ldots(53)$$

Equating to zero the imaginary part of eqn. (52) and using eqn. (53) we get

$$c_i\int_0^1 U\,[\,|DF|^2 + a^2|F|^2]\,dz =$$
$$\frac{R_1a}{2}\left[\int_0^1 U\,|D\Theta|^2\,dz - \tfrac{1}{2}\int_0^1 (D^2U)\,|\Theta|^2\,dz + a^2\int_0^1 U\,|\Theta|^2\,dz\right]. \qquad \ldots(54)$$

Since $R_1 > 0$ it clearly follows from eqn. (54) that under condition (i) of the theorem we must have

$$c_i > 0.$$

This proves the theorem.

Theorem 3 — If (i) $c_i = 0$ and U satisfies conditions of Remark 1 then

$$U_{min} < c_r < U_{max}.$$

Further $c_r > U_{min}$ or $c_r < U_{max}$ according as $U > 0$ and $D^2U < 0$ or $U < 0$ and $D^2U > 0$, $\forall\, z \in [0, 1]$.

(ii) $c_i < 0$ and $D^2U < 0$, $\forall\, z \in [0, 1]$ then $c_r > U_{min}$.

(iii) $c_i < 0$ and $D^2U > 0$, $\forall\, z \in [0, 1]$ then $c_r < U_{max}$.

PROOF : Using the transformations

$$w = (U - c)\,F$$

and $$\theta = (U - c^*)\,\Theta$$

eqns. (32) and (33) become

$$D\,[(U - c)^2\,DF] - a^2(U - c)^2\,F - R_2F - iR_1a(U - c^*)\,\Theta - Q(D^2 - a^2)\,F = 0 \quad \ldots(55)$$

and

$$[D^2 - a^2 - ia(U - c)]\,(U - c^*)\,\Theta = -\,(U - c)\,F \quad \ldots(56)$$

$$F = 0 = \Theta \text{ at } z = 0 \text{ and } z = 1.$$

Now proceeding exactly as in Theorem 1 it follows that

$$-\int_0^1 (U - c)^2\,[\,|DF|^2 + a^2\,|F|^2]\,dz - \int_0^1 R_2\,|F|^2\,dz + Q\int_0^1 [\,|DF|^2 + a^2\,|F|^2]\,dz = iR_1a\int_0^1 (U - c^*)\,\Theta F^* dz. \quad \ldots(57)$$

Also

Real part $$\left[\int_0^1 \Theta\,D^2\,[(U - c)\,\Theta^*]\,dz\right] = -\int_0^1 (U - c_r)\,|D\Theta|^2\,dz + \tfrac{1}{2}\int_0^1 (D^2U)\,|\Theta|^2\,dz. \quad \ldots(58)$$

Imaginary part of eqn. (57) equated to zero upon using eqns. (56) and (58) yields

$$\int_0^1 (U - c_r)[2c_i\{\,|DF|^2 + a^2\,|F|^2\} - R_1a\{\,|D\Theta|^2 + a^2\,|\Theta|^2\}]\,dz + \frac{R_1a}{2}\int_0^1 (D^2U)\,|\Theta|^2\,dz = 0. \quad \ldots(59)$$

If $c_i = 0$ and U satisfies conditions of Remark 1 so that $c = c_r \neq 0$ and $D^2U = 0$ everywhere in [0, 1], we have from eqn. (59)

$$R_1a\int_0^1 (U - c_r)\,[\,|D\Theta|^2 + a^2\,|\Theta|^2]\,dz = 0. \quad \ldots(60)$$

Equation (60) clearly implies that there exists a point $z = z_r \in [0, 1]$

s.t. $$[U - c_r]_{z=z_r} = 0.$$

This implies that

$$U_{min} < c_r < U_{max}.$$

Other results follow similarly from eqn. (59).

Theorem 4 — For small values of the coefficient of thermometric conductivity, $c_i < 0$ implies

$$U_{min} < c_r < U_{max}$$

Proof: Under the assumptions of the theorem the governing differential eqns. (32) and (33) become

$$D[(U-c)^2 DF] - a^2(U-c)^2 F - R_2 F - iR_1 a\theta - Q(D^2 - a^2) F = 0 \quad \ldots(61)$$

$$[a^2 + ia(U-c)]\,\theta = (U-c)\,F \quad \ldots(62)$$

where $F = \dfrac{w}{U-c}$ and $F = 0 = \theta$ at $z = 0$ and $z = 1$. ...(63)

Multiplying eqn. (61) by F^*, integrating over the vertical range of z and upon using eqns. (62) and (63) we get

$$-\int_0^1 (U-c)^2 [\,|DF|^2 + a^2|F|^2]\,dz - \int_0^1 R_2 |F|^2\,dz$$

$$+ Q\int_0^1 [\,|DF|^2 + a^2|F|^2]\,dz$$

$$- iR_1 a\left\{\int_0^1 \frac{a^2(U-c)\,|\theta|^2}{|(U-c)|^2}\,dz - ia\int_0^1 |\theta|^2\,dz\right\} = 0. \quad \ldots(64)$$

Imaginary part of eqn. (64) equated to zero yields

$$\int_0^1 (U-c_r)\left\{2c_i[\,|DF|^2 + a^2|F|^2] - \frac{R_1 a^3}{|(U-c)|^2}|\theta|^2\right\}dz = 0. \quad \ldots(65)$$

Since $c_i < 0$ and $R_1 > 0$, eqn. (65) clearly implies that there exist a point

$z = z_s \in [0, 1]$ s.t.

$$[U - c_r]_{z=z_s} = 0$$

This implies that

$$U_{min} < c_r < U_{max}.$$

Theorem 5 — The stability of generalized thermal shear flows is not influenced by the presence of magnetic field in the transverse direction for disturbances with $k = k_x$.

PROOF : When the impressed magnetic field is transverse to the direction of streaming the perturbation equations now become

$$i\rho_0 u\,[n + k_x U] + \rho_0 w \frac{dU}{dz} = - ik_x \delta p + \frac{\mu_e H}{4\pi}\,[ik_y h_x - ik_x h_y] \qquad \ldots(66)$$

$$i\rho_0 (n + k_x U)\, v = - ik_y \delta p \qquad \ldots(67)$$

$$i\rho_0 (n + k_x U)\, w = - \frac{d}{dz}(\delta p) - g\delta\rho + g\alpha\rho_0\theta + \frac{\mu_e H}{4\pi}\,[ik_y h_z - Dh_y] \ldots(68)$$

$$i\delta\rho(n + k_x U) = - \rho_0 w \frac{df}{dz} \qquad \ldots(69)$$

$$i(n + k_x U)\,\theta - \beta w = \kappa(D^2 - k^2)\,\theta \qquad \ldots(70)$$

$$i(n + k_x U)\, h_x = h_z DU + ik_y H u \qquad \ldots(71)$$

$$i(n + k_x U)\, h_y = ik_y H v \qquad \ldots(72)$$

$$i(n + k_x U)\, h_z = ik_y H w \qquad \ldots(73)$$

$$i(k_x h_x + k_y h_y) = - Dh_z \qquad \ldots(74)$$

$$i(uk_x + vk_y) = - Dw. \qquad \ldots(75)$$

The above system of equations can be reduced to the following equations (c.f. Chandrasekhar 1961)

$$(n + k_x U)(D^2 - k^2)\, w - k_x (D^2 U)\, w = gk^2 \left[\frac{wDf}{n + k_x U} + i\alpha\theta\right]$$

$$+ k_y^2 \frac{\mu_e H^2}{4\pi\rho_0} \left\{D\left(\frac{Dw}{n + k_x U}\right) - \frac{k^2 w}{n + k_x U}\right\}$$

$$- k_x k_y^2 \frac{\mu_e H^2}{4\pi\rho_0} D\left[\frac{DU}{(n + k_x U)^2}\, w\right] \qquad \ldots(76)$$

$$i(n + k_x U)\,\theta - \beta w = \kappa(D^2 - k^2)\,\theta. \qquad \ldots(77)$$

We observe that eqns. (76) and (77) coincide exactly with the hydrodynamic equations of generalized thermal shear flows when $k_y = 0$ (Gupta *et al.* 1977)

This proves the theorem.

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