ON INDUCED AND INTRINSIC CONNECTIONS OF RANDERS' HYPERSURFACE

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In this paper we have compared the induced and intrinsic connection coefficients of a Randers' hypersurface and thus certain conditions have been obtained.

INTRODUCTION

The n-dimensional Finsler space whose fundamental function $L(x, y)$ is given by

$$L(x, y) = (g_{ij}(x)\, y^i y^j)^{1/2} + b_i y^i, \; (y^i = \dot{x}^i)$$

is called a Randers' space. Such a space was first introduced by Randers (1941) and was named so by Ingarden (1957). This space has been studied from the standpoint of physics and geometry by many physicists and mathematicians. Matsumoto (1974) studied a Randers' space equipped with a general vector b_i in an ingenious manner and gave many fruitful results. The concerete forms of connection coefficients and curvature tensors of the Randers' space have been given by Shibata *et al.* (1977). The induced and intrinsic theories of subspaces of a Finsler space have been studied by Davies (1945) and Rund (1959, 1965).

The purpose of the present paper is to study the induced and intrinsic theory of hypersurface of a Randers' space. The connection coefficients of Randers' hypersurface can be expressed as sum of Riemannian Christoffel symbol and a third order tensor. Hence the idea to compare induced and intrinsic connection coefficients of a Randers' hypersurface is derived from the fact that the induced and intrinsic Christoffel symbols of a Riemannian hypersurface are equivalent and thus certain conditions have been obtained.

1. PRELIMINARIES

Let $\overset{f}{V}_n (n \geqslant 2)$ be an n-dimensional Randers' space whose metric function $L(x, y)$ $(y^i = \dot{x}^i)$ is given by

$$L(x, y) = (g_{ij}(x)\, y^i y^j)^{1/2} + b_i(x)\, y^i \qquad ...(1.1)$$

where $g_{ij}(x)$ is a Riemannian metric tensor of class C^2 and $b_i(x)$ is 1-form of class C^2 depending on the position only. We denote by $\overset{r}{V}_n$ the Riemannian space with the tensor $g_{ij}(x)$.

The connection coefficient of Cartan (Shibata *et al.* 1977) is given by

$$\Gamma^{*i}_{jk} = \left\{ {i \atop jk} \right\} + D^i_{jk} \qquad \text{...(1.2)}$$

where $\left\{ {i \atop jk} \right\}$ are the Christoffel symbols of second kind in $\overset{r}{V}_n$ and the difference tensor D^i_{jk} (Matsumoto 1974, Shibata *et al.* 1977) is given by

$$\begin{aligned} D^i_{jk} &= p^i b_{(jk)} + (h^i_j b_{ok} + h^i_k b_{oj} - h_{jk} b_{os} G^{is})/2 \\ &- \lambda A^i_{jk} + G^{is}(b_{[sj]}\, p_k + b_{[sk]}\, p_j) \\ &+ g^{ml}\, [G^{is}(b_{[ls]} + b_{[lo]}\, l_s \tau)\, A_{jkm} - A^i_{km}\,(b_{[lj]} + b_{[lo]}\, l_j \tau) \\ &- A^i_{jm}\,(b_{[lk]} + b_{[lo]}\, l_k \tau)]/\tau. \end{aligned} \qquad \text{...(1.3)}$$

The quantities $p^i (= y^i/L)$, h_{ij}, b_{ij}, G^{ij} etc., in (1.3) have their usual meanings as defined by Shibata *et al.* (1977) and the index o means the contraction by the unit vector p^i.

2. The Projection Factors

Consider a hypersurface $\overset{f}{V}_{n-1}$ of the Randers' space $\overset{f}{V}_n$ represented parametrically by the equations

$$x^i = x^i(u^\alpha) \; (i = 1, \ldots, n;\; \alpha = 1, \ldots, n-1) \qquad \text{...(2.1)}$$

in which the u^α denote the parameters of the hypersurface.**

It shall be assumed throughout that the functions (2.1) are of class C^3 and the matrix with entries

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} \qquad \text{...(2.2)}$$

has rank $n - 1$. For the sake of brevity we shall write

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta},\; B^{ij\ldots k}_{\alpha\beta\ldots\gamma} = B^i_\alpha\, B^j_\beta \ldots B^k_\gamma. \qquad \text{...(2.3)}$$

**In the sequel Latin and Greek indices run from 1 to n and 1 to $n - 1$ respectively. The summation convention, applied to both sets of indices, is used throughout.

With any vector $y^i(=\dot{x}^i)$ tangential to $\overset{f}{V}_{n-1}$ at some point with coordinates u^ϵ we may associate components v^α $(=\dot{u}^\alpha)$ in coordinate system of $\overset{f}{V}_{n-1}$ according to the relation

$$y^i = B^i_\alpha\,(u^\epsilon)\,v^\alpha. \qquad \text{...(2.4)}$$

The induced metric tensor of $\overset{f}{V}_{n-1}$ defined with respect to such a direction is given by

$$G_{\alpha\beta}(u, v) = G_{ij}(x, y)\,B^{ij}_{\alpha\beta} = \tfrac{1}{2}\,\frac{\partial^2 \overline{L}^2}{\partial v^\alpha \partial v^\beta} \qquad \text{...(2.5)}$$

where $\overline{L}$ is obtained from $L(x, y)$ by substituting from (2.1) and (2.4), that is

$$\overline{L}(u, v) = (g_{\alpha\beta}(u)\,v^\alpha v^\beta)^{1/2} + b_\alpha v^\alpha \qquad \text{...(2.6)}$$

in which $g_{\alpha\beta}$ is the metric tensor of the Riemannian hypersurface $\overset{r}{V}_{n-1}$ and b_α is defined by

$$b_\alpha = b_i B^i_\alpha\,. \qquad \text{...(2.7)}$$

In virtue of (1.1) and (2.6), we have the following lemma :

Lemma 1 — The hypersurface of a Randers' space is a Randers' space.

Further, we have

$$p^i = B^i_\alpha\,p^\alpha,\quad l_\alpha = B^i_\alpha\,l_i. \qquad \text{...(2.7')}$$

As usual, it is supposed that det $(G_{ij}) \neq 0$. Thus according to our assumption the tensor $G_{\alpha\beta}$ possesses an inverse $G^{\alpha\beta}(u, v)$ from which we construct the quantities (Rund 1959)

$$B^\alpha_i = G^{\alpha\epsilon} G_{ij} B^j_\epsilon \qquad \text{...(2.8)}$$

which satisfy the condition

$$B^\alpha_i\;B^i_\beta = \delta^\alpha_\beta\,. \qquad \text{...(2.9)}$$

A further useful set of identities is the following (Rund 1959) :

$$B^\alpha_i\;B^j_\alpha = \delta^j_i - N_i N^j \qquad \text{...(2.10)}$$

where the unit normal $N^i(x, y)$ is defined at each point P of $\overset{f}{V}_{n-1}$ with respect to the tangential direction y^j at P by the system of equations

$$N^i = G^{ij}(x, y)\, N_j,\ G_{ij}(x, y)\, N^i N^j = 1,\ N_i B^i_\alpha = 0 \qquad ...(2.11)$$

which in turn imply

$$N^i B^\alpha_i = 0. \qquad ...(2.12)$$

It has been shown (Rund 1959) that

$$G_{ij} = G_{\alpha\beta}\, B^\alpha_i\, B^\beta_j + N_i N_j,\ G^{ij} = G^{\alpha\beta} B^{ij}_{\alpha\beta} + N^i N^j. \qquad ...(2.13)$$

3. Relation Between Induced and Intrinsic Connection Coefficients

The Cartan's connection coefficients of the imbedding space $\overset{f}{V}_n$ are denoted by Γ^{*i}_{jk}. The induced connection coefficients of Randers' hypersurface $\overset{f}{V}_{n-1}$ are given by the relation (Rund 1959, 1965)

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i\, (B^i_{\beta\gamma} + \Gamma^{*i}_{jk}\, B^{jk}_{\beta\gamma}). \qquad ...(3.1)$$

In virtue of (3.1) and (1.2) we have

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i\, (B^i_{\beta\gamma} + \begin{Bmatrix} i \\ jk \end{Bmatrix} B^{jk}_{\beta\gamma}) + B^\alpha_i\, D^i_{jk}\, B^{jk}_{\beta\gamma}\,. \qquad ...(3.2)$$

Since the induced and intrinsic Christoffel symbols of the Riemannian hypersurface $\overset{r}{V}_{n-1}$ are equal, we have, from (3.2)

$$\Gamma^{*\alpha}_{\beta\gamma} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} + E^\alpha_{\beta\gamma} \qquad ...(3.3)$$

where

$$E^\alpha_{\beta\gamma} = B^\alpha_i\, D^i_{jk}\, B^{jk}_{\beta\gamma}. \qquad ...(3.4)$$

The expression (3.3) gives the induced connection coefficients of Randers' hypersurface $\overset{f}{V}_{n-1}$.

The intrinsic connection coefficients $'\Gamma^{*\alpha}_{\beta\gamma}$ are defined (Rund 1965) with respect to the metric $G_{\alpha\beta}$ of $\overset{f}{V}_{n-1}$ in a manner formally identical with the mode of definition of the coefficients Γ^{*i}_{jk} of $\overset{f}{V}_n$ in terms of the metric tensor G_{ij} of $\overset{f}{V}_n$. Writing down the relation corresponding to (1.2) for the hypersurface $\overset{f}{V}_{n-1}$ we have

$$'\Gamma^{*\alpha}_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - C^\alpha_{\beta\varepsilon}\, \Gamma^\varepsilon_{\delta\gamma}\, v^\delta = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} + D^\alpha_{\beta\gamma} \qquad ...(3.5)$$

where

$$C^{\alpha}_{\beta\varepsilon} = G^{\alpha\delta} C_{\beta\delta\varepsilon}, \; C_{\alpha\beta\gamma} = C_{ijk}\, B^{ijk}_{\alpha\beta\gamma} = \tfrac{1}{2} \frac{\partial^3 \bar{L}^2}{\partial v^{\alpha} \partial v^{\beta} \partial v^{\gamma}} \qquad \ldots(3.6)$$

and

$$D^{\alpha}_{\beta\gamma} = p^{\alpha} b_{(\beta\gamma)} + (h^{\alpha}_{\beta}\, b_{o\gamma} + h^{\alpha}_{\gamma}\, b_{o\beta} - h_{\beta\gamma} b_{o\delta} G^{\alpha\delta})/2$$

$$- {}'\lambda\, A^{\alpha}_{\beta\gamma} + G^{\alpha\delta}(b_{[\delta\beta]}\, p_{\gamma} + b_{[\delta\gamma]}\, p_{\beta})$$

$$+ g^{\lambda\mu}[G^{\alpha\delta}(b_{[\mu\delta]} + b_{[\mu o]}\, l_{\delta}\, \tau)\, A_{\beta\gamma\lambda} - A^{\alpha}_{\gamma\lambda}\, (b_{[\mu\beta]} + b_{[\mu o]}\, l_{\beta}\, \tau)$$

$$- A^{\alpha}_{\beta\lambda}\, (b_{[\mu\gamma]} + b_{[\mu o]}\, l_{\gamma}\, \tau)]/\tau \qquad \ldots(3.7)$$

in which

$${}'\lambda = b_{oo}/2 - b^{\alpha} b_{[\alpha o]}/\tau.$$

Denoting the difference of the induced and intrinsic connection coefficients of $\overset{f}{V}_{n-1}$ by $\Lambda^{\alpha}_{\beta\gamma}$ (Rund 1965), we have in virtue of (3.3) and (3.5)

$$\Lambda^{\alpha}_{\beta\gamma} = {}'\Gamma^{*\alpha}_{\beta\gamma} - \Gamma^{*\alpha}_{\beta\gamma} = D^{\alpha}_{\beta\gamma} - E^{\alpha}_{\beta\gamma}\,. \qquad \ldots(3.8)$$

Differentiating (2.7) covariantly with respect to u^{β} in $\overset{r}{V}_{n-1}$, we have

$$b_{\alpha\beta} = b_{ij}\, B^{i}_{\alpha}\, B^{j}_{\beta} + b_{i} \overset{r}{I}{}^{i}_{\alpha\beta}$$

where $\overset{r}{I}{}^{i}_{\alpha\beta}$ is the normal curvature vector of $\overset{r}{V}_{n-1}$. The above expression can be written as

$$b_{\alpha\beta} = b_{ij} B^{ij}_{\alpha\beta} + b_{i} \overset{r}{N}{}^{i} \overset{r}{\Omega}_{\alpha\beta} \qquad \ldots(3.9)$$

where $\overset{r}{N}{}^{i}$ is the unit normal at each point P of the Riemannian hypersurface $\overset{r}{V}_{n-1}$ and $\overset{r}{\Omega}_{\alpha\beta}$ are the coefficients of second fundamental form of $\overset{r}{V}_{n-1}$.

From (3.9), we have

$$b_{(\alpha\beta)} = b_{(ij)} B^{ij}_{\alpha\beta} + b_{i} \overset{r}{N}{}^{i} \overset{r}{\Omega}_{\alpha\beta}, \; b_{[\alpha\beta]} = b_{[ij]} B^{ij}_{\alpha\beta}\,. \qquad \ldots(3.10)$$

On direct calculation with the help of (2.7), (2.7′), (2.9), (2.10), (2.11), (2.13), (3.9) and (3.10), eqn. (3.8) reduces to

$$
\begin{aligned}
\Lambda^{\alpha}_{\beta\gamma} = {} & p^{\alpha} b_i \overset{r}{N}{}^i \overset{r}{\Omega}_{\beta\gamma} + \tfrac{1}{2} h^{\alpha}_{\beta}\, b_i \overset{r}{N}{}^i \overset{r}{\Omega}_{o\gamma} + \tfrac{1}{2} h^{\alpha}_{\gamma}\, b_i \overset{r}{N}{}^i \overset{r}{\Omega}_{o\beta} \\
& - \tfrac{1}{2} h_{\beta\gamma} b_i \overset{r}{N}{}^i \overset{r}{\Omega}_{o\delta} G^{\alpha\delta} - \tfrac{1}{2} A^{\alpha}_{\beta\gamma}\, b_i \overset{r}{N}{}^i \overset{r}{\Omega}_{oo} \\
& - \frac{1}{\tau} \overset{r}{N}{}^m \overset{r}{N}{}^l G^{\alpha\delta} B^{s}_{\delta}\, b_{[ls]}\, B^{jk}_{\beta\gamma}\, A_{jkm} - \overset{r}{N}{}^m \overset{r}{N}{}^l G^{\alpha\delta} b_{[lo]}\, l_s B^{jk}_{\beta\gamma}\, A_{jkm} \\
& + \frac{1}{\tau} \overset{r}{N}{}^m \overset{r}{N}{}^l b_{[lj]}\, B^{\alpha}_{i}\, B^{jk}_{\beta\gamma}\, A^{i}_{km} + \overset{r}{N}{}^m \overset{r}{N}{}^l b_{[lo]}\, l_j B^{\alpha}_{i}\, B^{jk}_{\beta\gamma}\, A^{i}_{km} \\
& + \frac{1}{\tau} \overset{r}{N}{}^m \overset{r}{N}{}^l b_{[lk]}\, B^{\alpha}_{i}\, B^{jk}_{\beta\gamma}\, A^{i}_{jm} + \overset{r}{N}{}^m \overset{r}{N}{}^l b_{[lo]}\, l_k B^{\alpha}_{i}\, B^{jk}_{\beta\gamma}\, A^{i}_{jm}. \qquad \text{...(3.11)}
\end{aligned}
$$

Though the above expression for the difference of induced and intrinsic connection coefficients for the Randers' hypersurface $\overset{f}{V}_{n-1}$ is cumbersome yet some interesting results can be deduced from it which we shall give in the following section.

4. Results

For a hypersurface of a Finsler space we have (Rund 1965)

$$\Lambda^{\alpha}_{\beta\gamma}\, v^{\beta} v^{\gamma} = 0 \qquad \text{...(4.1)}$$

Multiplying (3.11) by $v^{\beta}v^{\gamma}$, using (4.1) and the relations $A_{ijm}y^{j} = 0$, $h_{\alpha\beta}v^{\beta} = 0$, we obtain

$$b_i \overset{r}{N}{}^i \overset{r}{\Omega}_{\beta\gamma} v^{\beta} v^{\gamma} = 0.$$

Hence either $b_i \overset{r}{N}{}^i = 0$ or $\overset{r}{\Omega}_{\beta\gamma} v^{\beta} v^{\gamma} = 0$.

Case I — Let

$$b_i \overset{r}{N}{}^i \neq 0 \qquad \text{...(4.2)}$$

that is b_i is not tangential to the Riemannian hypersurface $\overset{r}{V}_{n-1}$, then

$$\overset{r}{\Omega}_{\beta\gamma} v^{\beta} v^{\gamma} = 0. \qquad \text{...(4.3)}$$

Hence we have the following theorem :

Theorem 4.1 — If b_i is not tangential to $\overset{r}{V}_{n-1}$, then the element of support y^i of $\overset{f}{V}_n$ are along the asymptotic direction with respect to the Riemannian hypersurface.

Let us assume that the Randers' space is a Landsberg space. Hence (Shibata *et al.* 1977)

$$b_{ij} = 0.$$

In virtue of this relation and (4.2), eqn. (3.9) gives

$$b_{\alpha\beta} \neq 0.$$

Hence we have the lemma :

Lemma 2 — The hypersurface of a Landsberg Randers' space is not, in general, a Landsberg space under the condition (4.2).

In virtue of the relations (3.11) and (4.3), we have the following theorem :

Theorem 4.2 — If the element of support of Randers' space $\overset{f}{V}_n$ are along the asymptotic direction with respect to Riemannian hypersurface, then a necessary and sufficient condition that the induced and intrinsic connection coefficients of the hypersurface of Landsberg Randers' space are equivalent is that

$$p^{\alpha} b_i \overset{r}{N}{}^i \overset{r}{\Omega}_{\beta\gamma} + \tfrac{1}{2} h^{\alpha}_{\beta}\, b_i \overset{r}{N}{}^i \overset{r}{\Omega}_{o\gamma} + \tfrac{1}{2} h^{\alpha}_{\gamma}\, b_i \overset{r}{N}{}^i \overset{r}{\Omega}_{o\beta}$$

$$- \tfrac{1}{2} h_{\beta\gamma} b_i \overset{r}{N}{}^i \overset{r}{\Omega}_{o\delta} G^{\alpha\delta} = 0. \qquad \text{...(4.4)}$$

Next, suppose that the vector b_i is irrotational, i.e.,

$$b_{[ij]} = 0.$$

In virtue of this relation and eqns. (4.3), (3.11), we have the following theorem :

Theorem 4.3 — If the element of support of $\overset{f}{V}_n$ are along the asymptotic direction of $\overset{r}{V}_{n-1}$ and if the vector b_i is irrotational then a necessary and sufficient condition that the induced and intrinsic connection coefficients of Randers' hypersurface be equivalent is that the relation (4.4) holds.

Case II — Let the element of support y^i of $\overset{f}{V}_n$ be not along the asymptotic direction of $\overset{r}{V}_{n-1}$. Then

$$b_i \overset{r}{N}{}^i = 0 \qquad \text{...(4.5)}$$

which implies that either $b_i = 0$ or b_i is tangential to $\overset{r}{V}_{n-1}$. Hence we have the following theorem :

Theorem 4.4 — If the element of support of $\overset{f}{V}_n$ is not in the asymptotic direction of the Riemannian hypersurface $\overset{r}{V}_{n-1}$, then either the space $\overset{f}{V}_n$ is Riemannian or b_i is tangential to $\overset{r}{V}_{n-1}$.

Next, suppose that b_i is tangential to the Riemannian hypersurface $\overset{r}{V}_{n-1}$. Then from (3.9) and (4.5) we obtain

$$b_{\alpha\beta} = 0 \text{ if and only if } b_{ij} = 0.$$

Thus we have the lemma :

Lemma 3 — If the vector b_i is tangential to $\overset{r}{V}_{n-1}$ then the Randers' hypersurface $\overset{f}{V}_{n-1}$ is a Landsberg space if and only if the imbedding Randers' space $\overset{f}{V}_n$ is Landsberg.

Further, we have the following :

Theorem 4.5 — If b_i is tangential to $\overset{r}{V}_{n-1}$ and if the Randers' space $\overset{f}{V}_n$ is Landsberg, then the induced and intrinsic connection coefficients of $\overset{f}{V}_{n-1}$ are equivalent and each is equal to the Riemannian Christoffel symbol of second kind with respect to $\overset{r}{V}_{n-1}$.

PROOF : We have $b_i\overset{r}{N}^i = 0$ and $b_{ij} = 0$.

Therefore (3.11) gives

$$\Lambda^{\alpha}_{\beta\gamma} = {}'\Gamma^{*\alpha}_{\beta\gamma} - \Gamma^{*\alpha}_{\beta\gamma} = 0$$

which is the first part of the theorem.

Since $b_{ij} = 0$, we have (Shibata *et al.* 1977) $D^i_{jk} = 0$,

which, in virtue of (3.3) and (3.4), gives

$${}'\Gamma^{*\alpha}_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix}. \text{ Hence the theorem.}$$

Finally, if the vector b_i is irrotational and tangential to $\overset{r}{V}_{n-1}$, then from (3.11) we have $\Lambda^{\alpha}_{\beta\gamma} = 0$.

Hence we get the following theorem :

Theorem 4.6 — The induced and intrinsic connection coefficients of Randers' hypersurface $\overset{f}{V}_{n-1}$ are equivalent provided the vector b_i be irrotational as well as tangential to the Riemannian hypersurface $\overset{r}{V}_{n-1}$.

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