A NUMBER OF SPINNING ROTORS MOUNTED ON A RIGID BODY

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The motion of a number of spinning rotors mounted on a rigid body has been analysed by deriving the vector equations of forces and moments. It has been established that a rigid body with spinning rotors fixed to it at their respective centroids dynamically behaves like a single rigid body if the centroid of the system of the spinning rotors coincides with that of the rigid body itself. The equations have been employed to investigate the motion of an aircraft carrying two identical propellors and finally a numerical example has been worked out.

INTRODUCTION

Miele (1962) studied the motion of a spinning rotor mounted on a rigid body and derived the equations of forces and moments where the rotor is symmetrical with respect to its axis of rotation and its angular velocity is constant both in its modulus and its orientation with respect to the rigid body. This paper deals with the motion of a number of spinning rotors mounted on a rigid body.

EQUATION OF FORCES

Let us consider two rotors which are fixed with respect to the rigid body at their respective centroids O_1 and O_2. $O_1X_1Y_1Z_1$ and $O_2X_2Y_2Z_2$ are two moving frames of reference fixed relative to the rigid body. O_1Z_1 and O_2Z_2 are the axes of symmetry for these two rotors respectively.

Let us suppose,

m_1 = mass of the 1st rotor

m_2 = mass of the 2nd rotor

m_0 = mass of the rigid body

$\vec{w}_{R_1}$ = constant relative angular velocity of the 1st rotor about its axis O_1Z_1.

$\vec{w}_{R_2}$ = constant relative angular velocity of the 2nd rotor about its axis O_2Z_2 of symmetry.

$\vec{w}$ = Instantaneous angular velocity of the rigid body with respect to a reference frame ΩPQR fixed in space.

$\dot{\vec{w}}$ = Derivative of $\vec{w}$ with respect to time

$(\hat{i}_1, \hat{j}_1, \hat{k}_1)$ = A set of unit vectors associated with the moving trihedral $O_1X_1Y_1Z_1$

$(\hat{i}_2, \hat{j}_2, \hat{k}_2)$ = A set of unit vectors associated with the moving trihedral $O_2X_2Y_2Z_2$.

The absolute acceleration of any point P_1 of the 1st rotor whose co-ordinates are (x_1, y_1, z_1) with respect to the moving system of axes $O_1X_1Y_1Z_1$, the origin O_1 being a point common to the rotor and the rigid body, can be given by

$$\vec{a}_1 = \vec{a}_{0_1} + \vec{a}_{r_1} + 2\vec{w} \times \vec{V}_{r_1} + \vec{w} \times (\vec{w} \times \overrightarrow{O_1P_1}) + \dot{\vec{w}} \times \overrightarrow{O_1P_1}$$

where $\vec{a}_{0_1}$ = absolute acceleration of the centroid O_1 of the rotor. $\vec{V}_{r_1}$ and $\vec{a}_{r_1}$ are the relative velocity and the relative acceleration of the point P_1 with reference to the moving frame $O_1X_1Y_1Z_1$.

Now we can calculate

$$\vec{V}_{r_1} = \vec{w}_{R_1} \times \overrightarrow{O_1P_1}$$

$$\vec{a}_{r_1} = -(x_1\hat{i}_1 + y_1\hat{j}_1)\,\dot{\phi}_1^2$$

where $\overrightarrow{O_1P_1} = \hat{i}_1x_1 + \hat{j}_1y_1 + \hat{k}_1z_1$ and $\dot{\phi}_1$ is the constant physical angular velocity of the 1st rotor about the axis O_1Z_1 of symmetry.

$$\vec{w}_{R_1} = \hat{k}_1\dot{\phi}_1$$

Similarly the absolute acceleration of any point P_2 of the 2nd rotor whose co-ordinates are (x_2, y_2, z_2) with respect to the system of axes $O_2X_2Y_2Z_2$ can be given by

$$\vec{a}_2 = \vec{a}_{0_2} + \vec{a}_{r_2} + 2\vec{w} \times \vec{V}_{r_2} + \vec{w} \times (\vec{w} \times \overrightarrow{O_2P_2}) + \dot{\vec{w}} \times \overrightarrow{O_2P_2}$$

where

$$\vec{V}_{r_2} = \vec{w}_{R_2} \times \overrightarrow{O_2P_2}$$

$$\vec{a}_{r_2} = -(x_2\hat{i}_2 + y_2\hat{j}_2)\,\dot{\phi}_2^2$$

give the relative velocity and the relative acceleration of the point P_2 respectively with reference to the moving frame $O_2X_2Y_2Z_2$.

$\dot{\phi}_2$ = constant physical angular velocity of the 2nd rotor with respect to the axis O_2Z_2 of symmetry.

$$\vec{w}_{R_2} = \hat{k}_2\dot{\phi}_2$$

$\vec{a}_{0_2}$ = absolute acceleration of the point O_2 of the rigid body.

Hence, we can write

$$\vec{a}_1 = \vec{a}_{0_1} - (x_1\hat{i}_1 + y_1\hat{j}_1)\,\dot{\phi}_1^2 + 2\vec{w} \times (\vec{w}_{R_1} \times \overrightarrow{O_1P_1}) + \vec{w} \times (\vec{w} \times \overrightarrow{O_1P_1}) + \dot{\vec{w}} \times \overrightarrow{O_1P_1}$$

$$\vec{a}_2 = \vec{a}_{0_2} - (x_2\hat{i}_2 + y_2\hat{j}_2)\,\dot{\phi}_2^2 + 2\vec{w} \times (\vec{w}_{R_2} \times \overrightarrow{O_2P_2}) + \vec{w} \times (\vec{w} \times \overrightarrow{O_2P_2}) + \dot{\vec{w}} \times \overrightarrow{O_2P_2}$$

Now let us consider a frame of reference $OXYZ$ relatively fixed with the rigid body having its origin at the point O which is also the centroid of the rigid body. Then the acceleration of any point P of the rigid body whose co-ordinates are (x, y, z) with respect to the frame of axes $OXYZ$ can be obtained as

$$\vec{a} = \vec{a}_0 + \vec{w} \times (\vec{w} \times \overrightarrow{OP}) + \dot{\vec{w}} \times \overrightarrow{OP}$$

Let $\vec{F}$ be the total external force acting on the system of the rigid body and the two rotors.

Then,

$$\vec{F} = \int_{m'} \vec{a}'\, dm$$

$$m' = m_1 + m_2 + m_0$$

where $\vec{a}'$ denotes the acceleration of the mass element dm at the point P' of the entire system.

Again,

$$\vec{a}' = \vec{a}_1 = \vec{a}_{0_1} + \vec{a}_{r_1} + 2\vec{w} \times \vec{V}_{r_1} + \vec{w} \times (\vec{w} \times \overrightarrow{O_1P_1}) + \dot{\vec{w}} \times \overrightarrow{O_1P_1}$$

in the mass m_1 of the 1st rotor,

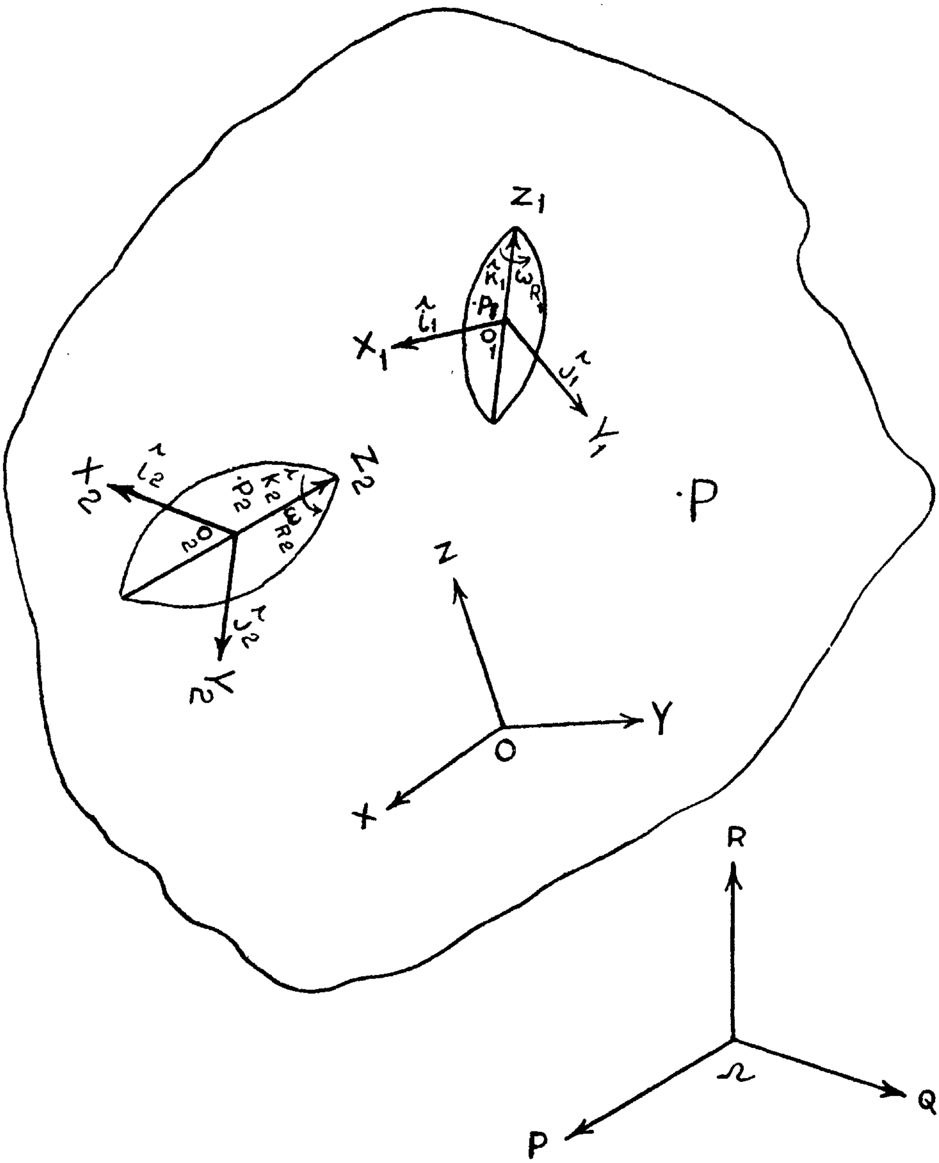

FIG. 1. Two spinning rotors mounted on a rigid body.

$$\vec{a}' = \vec{a}_2 = \vec{a}_{0_2} + \vec{a}_{r_2} + 2\vec{w} \times \vec{V}_{r_2} + \vec{w} \times (\vec{w} \times \overrightarrow{O_2P_2}) + \dot{\vec{w}} \times \overrightarrow{O_2P_2}$$

in the mass m_2 of the 2nd rotor,

$$\vec{a}' = \vec{a} = \vec{a}_0 + \vec{w} \times (\vec{w} \times \overrightarrow{OP}) + \dot{\vec{w}} \times \overrightarrow{OP}$$

in the mass m_0 of the rigid body,

where $\vec{a}_0$ is the absolute acceleration of the centroid of the rigid body. Hence, we have the equation of forces :

$$\vec{F} = \int_{m_1} \vec{a}_1 dm + \int_{m_2} \vec{a}_2 dm + \int_{m_0} \vec{a} dm$$

$$= \int_{m_1} \vec{a}_{0_1}\, dm - \dot{\phi}_1^2 \{ \int_{m_1} (x_1\hat{i}_1 + y_1\hat{j}_1)\, dm\} + 2\vec{w} \times (\vec{w}_{R_1} \times \int_{m_1} \overrightarrow{O_1P_1} dm)$$

$$+ \int_{m_2} \vec{a}_{0_2}\, dm - \dot{\phi}_2^2 \{ \int_{m_2} (x_2\hat{i}_2 + y_2\hat{j}_2) dm\} + 2\vec{w} \times (\vec{w}_{R_2} \times \int_{m_2} \overrightarrow{O_2P_2} dm)$$

$$+ \int_{m_0} \vec{a}_0 dm + \int_{m_1} \{\vec{w} \times (\vec{w} \times \overrightarrow{O_1P_1}) + \dot{\vec{w}} \times \overrightarrow{O_1P_1}\}\, dm$$

$$+ \int_{m_2} \{\vec{w} \times (\vec{w} \times \overrightarrow{O_2P_2}) + \dot{\vec{w}} \times \overrightarrow{O_2P_2}\}\, dm$$

$$+ \int_{m_0} \{\vec{w} \times (\vec{w} \times \overrightarrow{OP}) + \dot{\vec{w}} \times \overrightarrow{OP}\}\, dm. \qquad ...(1)$$

Since O_1, O_2 and O are the centroids of the two spinning rotors and the rigid body respectively, we have

$$\int_{m_1} \overrightarrow{O_1P_1} dm = \int_{m_2} \overrightarrow{O_2P_2} dm = \int_{m_0} \overrightarrow{OP} dm = 0$$

$$\int_{m_1} x_1 dm = \int_{m_1} y_1 dm = 0$$

$$\int_{m_2} x_2 dm = \int_{m_2} y_2 dm = 0. \qquad ...(2)$$

Using all these results in the above eqn. (1) of forces we get

$$\vec{F} = m_1\vec{a}_{0_1} + m_2\vec{a}_{0_2} + m_0\vec{a}_0 \qquad ...(3)$$

Again since the points O_1 and O_2 are fixed with respect to the rigid body,

$$\vec{a}_{0_1} = \vec{a}_0 + \vec{w} \times (\vec{w} \times \overrightarrow{OO_1}) + \dot{\vec{w}} \times \overrightarrow{OO_1}$$

$$\vec{a}_{0_2} = \vec{a}_0 + \vec{w} \times (\vec{w} \times \overrightarrow{OO_2}) + \dot{\vec{w}} \times \overrightarrow{OO_2} \qquad ...(4)$$

By means of these eqns. (4), the eqn. (3) reduces to the form

$$\vec{F} = m'\vec{a}_0 + \vec{w} \times \{\vec{w} \times (m_1 + m_2)\overrightarrow{OG}\} + \dot{\vec{w}} \times (m_1 + m_2)\overrightarrow{OG} \qquad ...(5)$$

where G is the centroid of the combination of two rotors.

If the point G coincides with the point O i.e. the centroid of the rigid body is the same as that of the combination of two rotors, the equation of forces reduces to the simple form

$$\vec{F} = m' \vec{a}_0 \qquad ...(6)$$

Hence the centroid of entire system of the two rotors and the rigid body moves with the absolute acceleration $\vec{a}_0$ given by eqn. (6).

If there be n number of spinning rotors fixed at their respective centroids O_1, O_2, O_3, ..., O_n with respect to the rigid body, $\vec{F}$ being the total external force acting on the whole system composed of the rotors and the rigid body,

$$\vec{F} = M'\vec{a}_0 + M_1 \{\vec{w} \times (\vec{w} \times \overrightarrow{OG}) + \dot{\vec{w}} \times \overrightarrow{OG}\} \qquad ...(7)$$

where M' is the total mass of the rotors and the rigid body, and M_1 the mass of the system of rotors.

Now if G coincides with O i.e. the centroid of the rigid body is the same as that of the system of rotors, the equation of forces is finally simplified to the form

$$\vec{F} = M' \vec{a}_0 \qquad ...(8)$$

Hence the centroid of the entire system of the rotors and the rigid body moves with the absolute acceleration given by eqn. (8) as if all the mass of the system were collected at it and as if all the external forces act at it.

Equation of Moments

The equation of moments with reference to the origin O of the body axes system in case of a system of the two rotors and the rigid body as described in the foregoing article under the heading "Equation of forces", can lead to the relationship

$$\vec{M} = \int_{m'} \overrightarrow{OP'} \times \vec{a}' \, dm$$

[using eqns. (4)]

$$= \int_{m'} \overrightarrow{OP'} \times \{\vec{a}_0 + \vec{w} \times (\vec{w} \times \overrightarrow{OP'}) + \dot{\vec{w}} \times \overrightarrow{OP'}\} \, dm$$

$$+ \int_{m_1} (\overrightarrow{OO_1} + \overrightarrow{O_1P_1}) \times (\vec{a}_{r_1} + 2\vec{w} \times \vec{V}_{r_1}) \, dm$$

$$+ \int_{m_2} (\overrightarrow{OO_2} + \overrightarrow{O_2P_2}) \times (\vec{a}_{r_2} + 2\vec{w} \times \vec{V}_{r_2}) \, dm. \qquad ...(9)$$

Now because of the relations (2),

$$\int_{m_1} \overrightarrow{OO_1} \times (\vec{a}_{r_1} + 2\vec{w} \times \vec{V}_{r_1})\, dm = \int_{m_2} \overrightarrow{OO_2} \times (\vec{a}_{r_2} + 2\vec{w} \times \vec{V}_{r_2})\, dm = 0 \quad ...(10)$$

Since the frame of reference $O_1X_1Y_1Z_1$ has its origin O_1 at the centroid of the rotor whose mass distribution is symmetrical about its axis of rotation O_1Z_1; O_1X_1, O_1Y_1 and O_1Z_1 must be its principal axes.

Hence,

$$\int_{m_1} z_1x_1dm = \int_{m_1} y_1z_1dm = \int_{m_1} x_1y_1dm = 0 \quad ...(11)$$

$$\int_{m_1} x_1^2\, dm = \int_{m_1} y_1^2\, dm. \quad ...(12)$$

If I_{R_1} be the moment of inertia of the 1st rotor about its axis of rotation,

$$I_{R_1} = \int_{m_1} (x_1^2 + y_1^2)\, dm$$

$$= 2 \int_{m_1} x_1^2\, dm = 2 \int_{m_1} y_1^2\, dm. \quad ...(13)$$

As consequence of the above relations,

$$\int_{m_1} \overrightarrow{O_1P_1} \times (\vec{a}_{r_1} + 2\vec{w} \times \vec{V}_{r_1})\, dm$$

$$= (\hat{i}_1w_2 - \hat{j}_1w_1)\, \dot{\phi}_1 I_{R_1}$$

$$= - I_{R_1} (\vec{w}_{R_1} \times \vec{w}) \quad ...(14)$$

where $\vec{w} = \hat{i}_1w_1 + \hat{j}_1w_2 + \hat{k}_1w_3$; w_1, w_2 and w_3 are the components of the angular velocity of the rigid body about the axes O_1X_1, O_1Y_1 and O_1Z_1 of the moving frame $O_1X_1Y_1Z_1$.

Now,

$$\int_{m'} \overrightarrow{OP'} \times \vec{a}_0 dm$$

$$= - \vec{a}_0 \times (m_1 + m_2)\, \overrightarrow{OG} \quad ...(15)$$

where G is the centroid of the combination of two rotors.

Now if the centroid G of the combination of two rotors coincides with that of the rigid body itself,

$$\int_{m'} \overrightarrow{OP'} \times \vec{a}_0 dm = 0 \qquad ...(16)$$

Finally the Equation of moments reduces to the form

$$\vec{M} + \vec{M}_{G_1} + \vec{M}_{G_2} = \int_{m'} \overrightarrow{OP'} \times \{\vec{w} \times (\vec{w} \times \overrightarrow{OP'}) + \dot{\vec{w}} \times \overrightarrow{OP'}\} dm \qquad ...(17)$$

where

$$\vec{M}_{G_1} = I_{R_1} (\vec{w}_{R_1} \times \vec{w})$$

$$\vec{M}_{G_2} = I_{R_2} (\vec{w}_{R_2} \times \vec{w})$$

represent the gyroscopic moments of the two rotors respectively about their axes of rotation.

In case of a number of rotors mounted on the rigid body as described earlier, we can show that

$$\vec{M} + \vec{M}_{G_1} + \vec{M}_{G_2} + \ldots + \vec{M}_{G_n}$$

$$= \int_{M'} \overrightarrow{OP'} \times \{\vec{w} \times (\vec{w} \times \overrightarrow{OP'}) + \dot{\vec{w}} \times \overrightarrow{OP'}\} dm \qquad ...(18)$$

where $\vec{M}_{G_1}, \vec{M}_{G_2}, \ldots, \vec{M}_{G_n}$ are their gyroscopic moments about their respective axes of rotation and P' is any point of integration belonging to the system of the rotors and the rigid body.

In fact the gyroscopic moment arises due to the coriolis acceleration associated with the motion of the rotor relative to the rigid body and is always present in vehicles containing rotating machinery (propellors, turbines, compressors, turbo-pumps etc.).

It is interesting to observe that the inertia terms on the right-hand side of eqns. (8) and (18) are identical with those in the equations of a totally rigid body.

Finally this observation leads to the solidification principle :

The equations of the instantaneous motion of the system formed by a number of spinning rotors plus a rigid body are formally identical with those of a totally rigid body, provided that the rotors are fixed with respect to the rigid body at their respective centroids in such a manner that the centroid of the combination of the

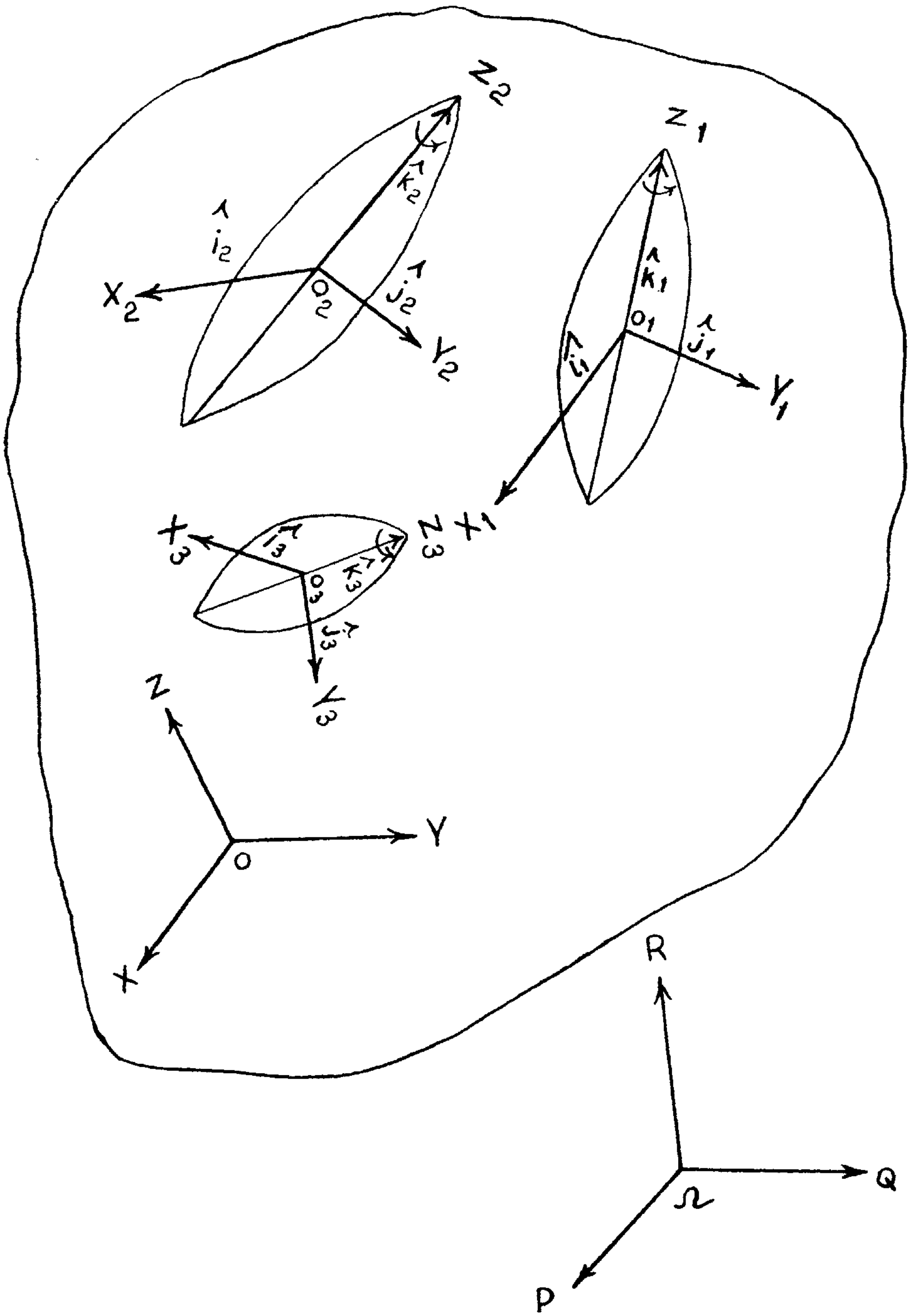

FIG. 2. A number of spinning rotors mounted on a rigid body.

rotors coincides with that of the rigid body itself and also that the following fictitious system of external forces is considered :

The actual forces and the gyroscopic forces.

Applicability of Solidification Principle

Equations (8) and (18) can be best suited to the general motion of an aircraft having rotating machines. Let us study the flight of an aircraft carrying two propellors. The following assumptions have been employed in regard to the design of the aircaft.

(1) The C.G. of the aircraft as a whole coincides with that of the combination of the propellors.

(2) Its C.G. is the origin of the body axes system $OXYZ$ which is defined as follows :

X-axis is contained in the plane of symmetry of the aircraft and is positive forward; the Z-axis is perpendicular to the X-axis, contained in the plane of symmetry, and positive downwards for normal flight attitude; the Y-axis is perpendicular to the plane of symmetry and is directed in such a way that the trihedral $OXYZ$ is right-handed.

$(\hat{i}, \hat{j}, \hat{k})$ = a set of unit vectors associated with the body axes.

The body axes are principal and the moment of inertia (B) of the entire aircraft about Y-axis is approximately the same as that (C) about Z-axis.

(3) The two propellors are identical in all respects. The axis of rotation of each propellor is parallel to the longitudinal axis (X-axis) of the aircraft. The propellors are fixed with respect to the aircraft body at their respective centroids and are also symmetrical about their respective axes of rotation. The mass distribution of each propellor is symmetrical with respect to its axis of rotation.

The controls of the flight are manipulated in such a way that the total propellor thrust and the aerodynamic force act at its C.G. The total propellor thrust is maintained along the X-axis. Here we shall deal with the class of flight paths flown with the total propellor thrust balanced by the drag and side force at all time instants. This type of path is called conservative path. Since this analysis is concerned with trajectories characterized by short ranges and/or velocities which are small with respect to the escape velocity, the Earth can be regarded as ideally flat and nonrotating.

In the light of the foregoing theory, the general equations of motion of the aircraft are given by

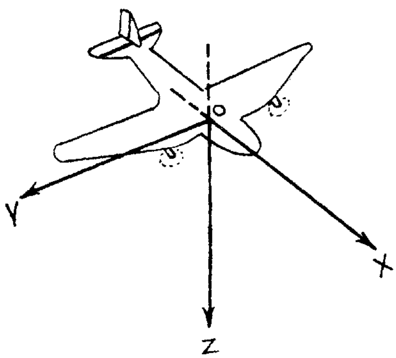

FIG. 3. Body axes system associated with an aircraft.

$$\vec{T} + \vec{W} + \vec{A} = \frac{W}{g} \vec{a} \qquad \text{...(19)}$$

$$\vec{M}_{G_1} + \vec{M}_{G_2} = \int_{w/g} [\overrightarrow{OP'} \times \{\vec{w} \times (\vec{w} \times \overrightarrow{OP'}) + \dot{\vec{w}} \times \overrightarrow{OP'}\}]\, dm \qquad \text{...(20)}$$

For conservative path,

$$\vec{T} + \vec{D} + \vec{Q} = 0 \qquad \text{...(21)}$$

gives the constriant.

where $\vec{T}$ is the thrust, $\vec{A}$ the aerodynamic force, $\vec{W}$ the gravitational force, $\vec{Q}$ the side force, $\vec{D}$ the drag, $\vec{a}$ the acceleration of the C.G. of the aircraft, g the acceleration due to gravity and t the time.

Furthermore, $\vec{M}_{G_1}$ and $\vec{M}_{G_2}$ denote the gyroscopic moments of the two propellors respectively.

$$\vec{M}_{G_1} = I_{R_1} (\vec{w}_{R_1} \times \vec{w}) = \vec{M}_{G_2}$$

$$\vec{w}_{R_1} = \hat{i} w_{R_1}$$

I_{R_1} and w_{R_1} are the moment of inertia and the constant angular velocity of either identical propellor about its axis of rotation.

Equations (19), (20) and (21) lead to the scalar relationships (22) and (23) in the tangent-normal directions and on the body axes respectively.

$$\dot{X} - V\cos\theta = 0$$

$$\dot{h} - V\sin\theta = 0$$

$$\dot{V} + g\sin\theta = 0$$

$$\dot{\theta} - \frac{g}{V}\left(\frac{L}{W} - \cos\theta\right) = 0 \qquad \ldots(22)$$

$$B\dot{w}_2 - (B - A)\, w_1 w_3 + 2I_{R_1}\, w_{R_1}\, w_3 = 0$$

$$B\dot{w}_3 - (A - B)\, w_1 w_2 - 2I_{R_1}\, w_{R_1}\, w_2 = 0$$

$$A\dot{w}_1 = 0 \qquad \ldots(23)$$

where X denotes the horizontal distance, h the altitude, V the velocity, θ the path inclination to the horizontal, W the gross weight of the aircraft, L the lift and the dot sign a derivative with respect to time t. w_1, w_2, w_3 are the components of the angular velocity of the aircraft along the body axes. A, B, C are the principal and central moments of inertia of the aircraft ($C = B$). Equations (22) and (23) are respectively associated with its translational and rotational motion. The ratio $n = L/W$ is called load factor. The case where the load factor is constant and equal to two and the C.G. of the aircraft moves in a vertical plane is discussed.

In order to simplify the analysis let us select the path inclination as the new independent variable and introduce the dimensionless co-ordinates

$$X' = \frac{Xg}{V_i^2}, \quad h' = \frac{hg}{V_i^2}, \quad u = \frac{V}{V_i} \qquad \ldots(24)$$

where V_i is the initial velocity.

In this way, the differential equations which can be obtained out of eqns. (22) are written as

$$\frac{dX'}{d\theta} = \frac{u^2\cos\theta}{2 - \cos\theta}$$

$$\frac{dh'}{d\theta} = \frac{u^2\sin\theta}{2 - \cos\theta}$$

$$\frac{du}{d\theta} = \frac{-u\sin\theta}{2 - \cos\theta}. \qquad \ldots(25)$$

Let us assume the initial conditions

$$t_i = \theta_i = X'_i = h'_i = 0, \quad u_i = 1 \qquad \ldots(26)$$

in consideration of which the particular solutions of the eqns. (25) can be obtained as

$$u = \frac{1}{2 - \cos\theta} \qquad \ldots(27)$$

$$X' = \frac{(3 - \cos\theta)\sin\theta}{3(2 - \cos\theta)^2} + \frac{2}{3\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{\theta}{2}\right) \qquad \ldots(28)$$

$$h' = \frac{(3 - \cos\theta)(1 - \cos\theta)}{2(2 - \cos\theta)^2}. \qquad \ldots(29)$$

Using eqns. (27) and the third of the relations (24) in the 4th of eqns. (22), we have

$$\frac{dt}{d\theta} = \frac{V_i}{g(2 - \cos\theta)^2} \qquad \ldots(30)$$

and because of the initial conditions (26) it leads to the particular solution

$$t = \frac{V_i}{g}\left[\frac{\sin\theta}{3(2 - \cos\theta)} + \frac{4}{3\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{\theta}{2}\right)\right]. \qquad \ldots(31)$$

It can be noticed that the trajectory is a loop and the velocity at its highest point ($\theta = \pi$) is $u = 1/3$. The co-ordinates of the highest point are given by

$$X'_A = \frac{\sqrt{3}\pi}{9}$$

$$h'_A = \tfrac{4}{9}. \qquad \ldots(32)$$

The velocity at the point $\left(\theta = \frac{\pi}{2}\right)$ where the aircraft climbs up vertically is $u' = \frac{1}{2}$.

So as to discuss the rotational motion of the aircraft let us suppose that it is set spinning with an angular velocity w_0 about a line passing through its C.G., making an angle α to the vertical and lying in the plane of symmetry while the direction of the flight is just vertical $\left(\theta = \frac{\pi}{2}\right)$. Its angular velocity at a subsequent time t is

$$\vec{w} = \hat{i}w_1 + \hat{j}w_2 + \hat{k}w_3.$$

At $\quad t = 0,$

$$w_2 = 0, \quad w_3 = w_0\sin\alpha, \quad w_1 = w_0\cos\alpha. \qquad \ldots(33)$$

In consequence of the last of eqns. (23), w_1 remains constant.

Multiplying the 2nd in eqns. (23) by $\sqrt{-1}$ and adding the result to the 1st of these equations, we have

$$B\dot{S} - \sqrt{-1}\,[(A - B)w_1 + 2I_{R_1}w_{R_1}]\,S = 0$$

where $S = w_2 + \sqrt{-1}\, w_3$

The general solution is

$$S = S_0 \exp(\sqrt{-1}\, Kt) \qquad \text{...(34)}$$

where

$$K = \frac{(A - B) w_1 + 2I_{R_1} w_{R_1}}{B}. \qquad \text{...(35)}$$

From the initial conditions (33), we can express

$$w_2 = - w_0 \sin\alpha \sin Kt, \quad w_3 = w_0 \sin\alpha \cos Kt$$

$$w_1 = w_0 \cos\alpha \qquad \text{...(36)}$$

which reveals that the physical angular velocity of the aircraft remains constant and equal to its initial magnitude w_0, which can be readily shown from the eqns. (23) and that the spin components w_2 and w_3 not only depend on the mass distribution of the entire aircraft about the body axes but also on the spins (w_{R_1}) of the propellors.

But the time t is measured from the instant at which the aircraft takes up the vertical position. Now the time taken by it to reach the highest point A from its vertical position B is given by [using eqn. (31)]

$$t' = \frac{V_i}{g}\left(\frac{2\pi}{9\sqrt{3}} - \frac{1}{6}\right) \qquad \text{...(37)}$$

Hence, the components of the angular velocity at the highest point of the flight path can be expressed as

$$w_2 = - w_0 \sin\alpha \sin\left\{\frac{KV_i}{g}\left(\frac{2\pi}{9\sqrt{3}} - \frac{1}{6}\right)\right\}$$

$$w_3 = w_0 \sin\alpha \cos\left\{\frac{KV_i}{g}\left(\frac{2\pi}{9\sqrt{3}} - \frac{1}{6}\right)\right\}$$

$$w_1 = w_0 \cos\alpha. \qquad \text{...(38)}$$

It can be noted that all the points of the aircraft body except its C.G., which traces the loop, travel along helical curves during the flight from the position B. It is interesting to note further that if the aircraft carries a counter rotating propellor as third one, it gives rise to gyroscopic moment of opposite sign with respect to those of the first two propellors and hence reduces the gyroscopic effect.

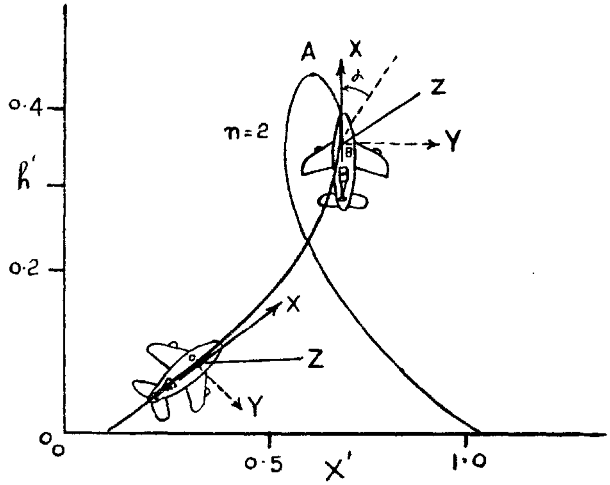

FIG. 4. Trajectory of the C.G. of the aircraft.

NUMERICAL EXAMPLE

Let us take the case of an aircraft having

$$V_i = 88.6 \text{ mts./sec.}, \quad W = 4000 \text{ Kg. wt.},$$

$$I_{R_1} = I_{R_2} = 20.8 \text{ Kg. mts}^2,$$

$$C = B = 3.2 \times 10^3 \text{ Kg. mts}^2, \quad A = 4.8 \times 10^3 \text{ Kg. mts}^2.,$$

$$w_{R_1} = \dot{w}_{R_2} = 28 \text{ revolutions/sec.},$$

$$w_\theta = \tfrac{1}{2} \text{ radians/sec.}, \quad \alpha = \frac{\pi}{3} \text{ radians},$$

$$g = 9.8 \text{ mts./sec}^2,$$

and this data gives $K = 2.4$ radians/sec.

The horizontal distance, the altitude, the velocity and the components of the angular velocity i.e. the complete attitude of the aircraft at the highest point A can be given by

$$X_A = \frac{X'_A V_i^2}{g} = 481.8 \text{ mts.}$$

$$h_A = \frac{h'_A V_i^2}{g} = 354.4 \text{ mts.}$$

$$V_A = \frac{V_i}{3} = 29.5 \text{ mts./sec.}$$

$$t' = 2.1 \text{ sec.}$$

$$w_2 = -\tfrac{1}{2} \sin \frac{\pi}{3} \sin (5.04 \text{ rads.}) = 0.41 \text{ rads./sec.}$$

$$w_3 = \tfrac{1}{2} \sin \frac{\pi}{3} \cos (5.04 \text{ rads.}) = 0.14 \text{ rads./sec.}$$

$$w_1 = \tfrac{1}{2} \cos \frac{\pi}{3} = 0.25 \text{ rads./sec.}$$

GYROSCOPIC COUPLE

Let us consider an airplane which has a rotary engine rotating in a clockwise direction when viewed from behind. The airplane makes a left turn and its mass centre describes a circular arc C in a horizontal plane.

The gyroscopic couple which comes into play is

$$\vec{G} = I_R(\vec{w}_R \times \vec{w})$$

where I_R and $\vec{w}_R$ are the moment of inertia and the angular velocity of the rotary engine about its axis of rotation which is also the axis of symmetry with respect to its geometry and mass distribution and is parallel to the tangent to the flight path, and $\vec{w}$ the angular velocity of the airplane.

If O be a fixed point on the axis of the rotor, along which is a unit vector $\hat{i}$,′ $\hat{k}'$ in the vertical direction and $\hat{j}'$ completes the triad $(\hat{i}', \hat{j}', \hat{k}')$,

$$\vec{w}_R = \hat{i}' w_R; \quad \vec{w} = \hat{k}' w$$

$$\vec{G} = -\, I_R w_R w \hat{j}'$$

where w_R and w are the physical angular velocities of the rotor and the aircraft about the axis of the former and the vertical respectively. It is clear that the gyroscopic couple of magnitude $I_R w_R w$, which is produced by a pair of gyroscopic forces in the rotating plane of $\hat{i}'$ and $\hat{k}'$, tends to make the nose of the airplane rise.

Hence, to maintain the steady flight in the horizontal circle, the pilot must set the rudder and elevator in such a way that the aerodynamic forces acting on them produce the balancing couple. If the pilot flies the airplane with a rotary engine in the same way as he would fly a similar airplane with a stationary engine, the couple required for the afore-said steady flight will not be available. Hence, in absence of the balancing couple, the nose will rise. Thus gyroscopic forces affect the control characteristics.

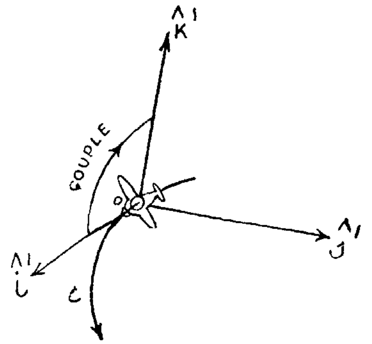

FIG. 5. Gyroscopic couple on the airplane turning.

ELASTIC DEFORMATION OF ROTOR

Significant stresses are produced in a large rotor rotating at high speed. An important instance is the design of a rotor mounted on a helicopter. Large stresses are produced due to centrifugal and aerodynamic forces. Hence, regarding the centrifugal forces as the body forces and the aerodynamic forces as the surface forces, the equilibrium and compatibility equations can be formed. Finally, the stress distribution and the state of strain in case of a rotor can be studied in detail by solving these equations with the prescribed boundary conditions.

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