CONFORMALLY FLAT NON-STATIC PLANE SYMMETRIC PERFECT FLUID DISTRIBUTIONS

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Solutions of Einstein's field equations representing plane symmetric perfect fluid distribution which are conformally flat have been derived. Various physical and geometrical properties of the models have also been discussed.

1. INTRODUCTION

Conformally flat space-times are of particular interest in view of their degeneracy in the context of Petrov classification. A number of physically significant metrics are conformally flat like Schwarzschild internal solution and Lemaitre universe. A spherically symmetric non-static generalization of Schwarzschild's internal solution was obtained by Singh and Abdussattar (1974). In this paper, we obtain non-static solutions of plane symmetry which are conformally flat and represent perfect fluid distribution. The explicit expressions for pressure, density, expansion, rotation, shear and the non-vanishing components of flow vector have also been obtained.

We consider the metric in the form

$$ds^2 = A^2(dx^2 - dt^2 + dy^2 + dz^2) \qquad ...(1)$$

where the metric potentials are functions of x and t. The energy momentum tensor for perfect fluid distribution is given by

$$T_i^j = (\epsilon + p)\, v_i v^j + p g_i^j \qquad ...(2)$$

together with

$$g_{ij} v^i v^j = -1 \qquad ...(3)$$

p being the pressure, ϵ the density and v^i the flow vector satisfying (3). The field equations are

$$-8\pi T_i^j = R_i^j - \tfrac{1}{2} R g_i^j + \Lambda \delta_i^j . \qquad ...(4)$$

From equations (2) and (4), we find that

$$T_1^2 = T_1^3 = T_2^3 = T_2^4 = T_3^4 = 0.$$

Hence $v_2 = v_3 = 0$. The remaining field equations are

$$8\pi\{(\epsilon + p)\ v_1^2 + pA^2\} = \frac{3A_1^2}{A^2} + \frac{A_4^2}{A^2} - \frac{2A_{44}}{A} - \Lambda A^2 \qquad ...(5)$$

$$8\pi pA^2 = \frac{2A_{11}}{A} - \frac{A_1^2}{A^2} - \frac{2A_{44}}{A} + \frac{A_4^2}{A^2} - \Lambda A^2 \qquad ...(6)$$

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$$8\pi\{(\epsilon + p)\ v_4^2 - pA^2\} = \frac{A_1^2}{A^2} - \frac{2A_{11}}{A} + \frac{3A_4^2}{A^2} + \Lambda A^2 \qquad ...(8)$$

$$8\pi\{(\epsilon + p)\ v_1 v_4\} = \frac{4A_1A_4}{A^2} - \frac{2A_{14}}{A}. \qquad ...(9)$$

The equation (3) reduces to

$$v_1^2 - v_4^2 = - A^2. \qquad ...(10)$$

2. Solution of the Field Equations

From (5) and (6), we have

$$8\pi(\epsilon + p)\ v_1^2 = \frac{4A_1^2}{A^2} - \frac{2A_{11}}{A}. \qquad ...(11)$$

From (6) and (8), we have

$$8\pi(\epsilon + p)\ v_4^2 = \frac{4A_4^2}{A^2} - \frac{2A_{44}}{A}. \qquad ...(12)$$

Equations (11) and (12) lead to

$$8\pi\{(\epsilon + p)\ (v_1^2 - v_4^2)\} = \frac{4A_1^2}{A^2} - \frac{2A_{11}}{A} - \frac{4A_4^2}{A^2} + \frac{2A_{44}}{A}. \qquad ...(13)$$

From eqns. (10) and (13), we have

$$8\pi(\epsilon + p)\ A^2 = \frac{2A_{11}}{A} - \frac{4A_1^2}{A^2} + \frac{4A_4^2}{A^2} - \frac{2A_{44}}{A}. \qquad ...(14)$$

Equations (6) and (14) lead to

$$8\pi\epsilon = \frac{3}{A^2}\left[\frac{A_4^2}{A^2} - \frac{A_1^2}{A^2}\right] + \Lambda. \qquad ...(15)$$

From eqn. (9), we have

$$64\pi^2(\epsilon + p)^2\ v_1^2\ v_4^2 = \frac{16A_1^2\ A_4^2}{A^4} + \frac{4A_{14}^2}{A^2} - \frac{16A_{14}}{A}.\frac{A_1A_4}{A^2}. \qquad ...(16)$$

Also from (11) and (12), we have

$$64\pi^2(\epsilon + p)^2 v_1^2 v_4^2 = \frac{16A_1^2 A_4^2}{A^4} - \frac{8A_1^2}{A^2}\cdot\frac{A_{44}}{A} - \frac{8A_{11}}{A}\cdot\frac{A_4^2}{A^2} + \frac{4A_{11}}{A}\cdot\frac{A_{44}}{A}. \quad ...(17)$$

Equations (16) and (17) lead to

$$2AA_{44}A_1^2 + 2AA_{11}A_4^2 - A^2A_{11}A_{44} + A^2A_{14}^2 - 4AA_1A_4A_{14} = 0. \quad ...(18)$$

Putting $A = e^{\lambda}$ in (18), we have

$$\lambda_4^2 \lambda_{11} + \lambda_1^2 \lambda_{44} - \lambda_{11}\lambda_{44} + \lambda_{14}^2 - 2\lambda_1\lambda_4\lambda_{14} = 0. \quad ...(19)$$

Two cases arise :

Case (i) : $\lambda_{14} \neq 0$:

When we apply Monge's method in eqn. (19), we find the following solution

$$e^{-\lambda} = [\phi(t + mx) - nx] \quad ...(20)$$

where m, n are constants and ϕ is an arbitrary function of its argument.

The metric therefore reduces to the form

$$ds^2 = [\phi(t + mx) - nx]^{-2} [dx^2 - dt^2 + dy^2 + dz^2] \quad ...(21)$$

when $n = 0$, the metric (21) transforms to Robertson-Walker metric with zero curvature.

Case (ii) :

$$\lambda_{14} = 0 \quad ...(22)$$

Equation (19) reduces to

$$\lambda_1^2 \lambda_{44} + \lambda_4^2 \lambda_{11} - \lambda_{11}\lambda_{44} = 0. \quad ...(23)$$

From eqn. (22), we have

$$\lambda = f(x) + g(t). \quad ...(24)$$

Equation (23) and (24) lead to

$$f_1^2 g_{44} + g_4^2 f_{11} - f_{11}g_{44} = 0. \quad ...(25)$$

From eqn. (25), we have

$$\frac{f_{11}}{f_1^2} = \frac{g_{44}}{g_{44} - g_4^2} = K \quad ...(26)$$

where K is a constant. From eqn. (26), we have

$$f = \log \{L(Kx + M)\}^{-1/K} \quad ...(27)$$

and

$$g = \log \left\{ P \left(\frac{Kt}{1-K} + N \right) \right\}^{(1-K)/K} \qquad \ldots(28)$$

where L, M, N and P are constants of integration. Hence

$$A = \left[\{L(Kx + M)\}^{-1/K} \left\{ P \left(\frac{Kt}{1-K} + N \right) \right\}^{(1-K)/K} \right]. \qquad \ldots(29)$$

By suitable transformations of coordinates, the metric reduces to the form

$$ds^2 = \beta^2 X^{-2/K}\, T^{(2-2K)/K} \,(dX^2 - dT^2 + dY^2 + dZ^2) \qquad \ldots(30)$$

where β^2 is an arbitrary constant.

3. Some Physical and Geometrical Features

The pressure and density for the model (21) are given by

$$8\pi p = 2\phi''(\phi - nx)(1 - m^2) - 3\phi'^2(1 - m^2) - 6\phi' mn + 3n^2 - \Lambda \qquad \ldots(31)$$

and

$$8\pi\epsilon = 3\phi'^2(1 - m^2) + 6\phi' mn - 3n^2 + \Lambda \qquad \ldots(32)$$

where a prime indicates differentiation with respect to the argument.

The non-vanishing components of flow vector are given by

$$v_1 = \frac{m}{\sqrt{1-m^2}} \cdot \frac{1}{(\phi - nx)} \qquad \ldots(33)$$

and

$$v_4 = \frac{1}{(\phi - nx)\sqrt{1-m^2}}. \qquad \ldots(34)$$

The reality condition $(\epsilon + p) > 0$ (Ellis 1971) and $v_1^2 > 0$ imply that $m^2 < 1$ and ϕ'' should be positive. The condition $(\epsilon + 3p) > 0$ leads to

$$\phi'' > \frac{\phi'^2}{(\phi - nx)} + \frac{2\phi' mn}{(\phi - nx)(1 - m^2)} + \frac{\Lambda - 3n^2}{3(\phi - nx)(1 - m^2)}. \qquad \ldots(35)$$

The non-vanishing components of the vector $\dot{v}_i = v_{i;j}v^j$ are

$$\dot{v}_1 = \frac{n}{(\phi - nx)(1 - m^2)} \qquad \ldots(36)$$

and

$$\dot{v}_4 = \frac{mn}{(\phi - nx)(1 - m^2)}. \qquad \ldots(37)$$

The flow is therefore non-geodetic in general. However when $n = 0$, it becomes geodetic. The expressions for expansion θ, rotation ω and shear tensor σ_{ij} are given by

$$\theta = \frac{3mn}{\sqrt{1-m^2}} + 3\phi' \sqrt{1-m^2} \quad ...(38)$$

$$\omega = 0 \quad ...(39)$$

$$\sigma_{ij} = 0. \quad ...(40)$$

The model is therefore expanding, non-rotating, non-shearing and non-geodetic in general.

The pressure and density for the model (30) are given by

$$8\pi p = \frac{X^{2/K}}{\beta^2 K^2 T^{(2-2K)/K}} \left[\frac{2K+1}{X^2} + \frac{(1-K)(3K-1)}{T^2}\right] - \Lambda \quad ...(41)$$

and

$$8\pi\epsilon = \frac{X^{2/K}}{\beta^2 K^2 T^{(2-2K)/K}} \left[\frac{3(1-K)^2}{T^2} - \frac{3}{X^2}\right] + \Lambda. \quad ...(42)$$

The non-vanishing components of flow vector are given by

$$v_1^2 = \frac{\beta^2 X^{-2/K} T^{2/K}}{(X^2 - T^2)} \quad ...(43)$$

and

$$v_4^2 = \frac{\beta^2 T^{(2-2K)/K} X^{(2K-2)/K}}{(X^2 - T^2)}. \quad ...(44)$$

The reality condition $(\epsilon + p) > 0$ (Ellis 1971) and $v_1^2 > 0$ lead to

$$K < 1. \quad ...(45)$$

The reality condition $(\epsilon + 3p) > 0$ leads to

$$\frac{3X^{(2-2K)/K} \{(1-K) X^2 + T^2\}}{\beta^2 K T^{2/K}} > \Lambda \quad ...(46)$$

which gives condition on Λ.

REFERENCES

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