SOME COUNTER-EXAMPLES FOR QUASINORMAL OPERATORS AND RELATED RESULTS

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Our main interest in this paper is to produce counter examples. In fact two counter examples are supplied to show that the famous Fuglede-Putnam-Rosenblum theorem need not hold for quasinormal operators and the following theorem is also disproved by means of an example.

Theorem — Let $T = UP$ be a quasinormal operator. If $\varepsilon > 0$, $C > 0$ are given then there exists $\delta > 0$ such that for some S in $B(H)$, $\| S \| \leqslant C$ and $\| PS - SP \| \leqslant \delta$ imply $\| T^*S - ST^* \| \leqslant \varepsilon$.

Besides studying some new properties of quasinormal operators, the concepts of skew-quasinormal and co-quasinormal operators are introduced.

Following open problem is raised :

"Whether two similar non-isometric quasinormal operators are unitarily equivalent ?"

1. INTRODUCTION

Let H be a complex Hilbert space. It is well known that "If M and N are normal operators on H and T is any other operator on H then $MT = TN$ implies $M^*T = TN^*$". This result is popularly known as Fuglede-Putnam-Rosenblum (FPR, for short) theorem. It is also known that this theorem does not hold for subnormal operators (Halmos 1967). Hence it is interesting to ask : Does the FPR-theorem hold for quasinormal operators? In Section 2 we give some counter-examples to show that the FPR-theorem fails for quasinormal operators in general.

Motivated by FPR-theorem, Moore (1975) discussed the problem: Is $N^*T - TN^*$ small whenever $NT - TN$ is small where N is a normal operator and T is any operator on H? In Section 3 we modify these considerations for quasinormal operators in order to supply a counter example in this context. We also modify the concepts of "almost-normal" and "near normal" due to Bastian and Harrison (1974) for unitary operators.

In the last section we obtain some interesting properties of quasinormal operators besides studying some new concepts.

The detailed discussion is postponed to respective sections.

2. Falsity of FPR-Theorem for Quasinormals

In order to discuss the invalidity of the FPR-theorem for quasinormal operators it is essential to split the problem into two parts. That is for those quasinormal operators that are isometries and for non-isometric quasinormal operators. It is clear that for every operator $T.T = T.T$ holds, and shifts are examples showing that $TT^* \neq T^*T$. This invariably demonstrates the falsity of FPR-theorem for proper isometries with $M = N = T$ in the formulation of the FPR-theorem that is given in the introduction. The very example of shifts shows that when $M = N$ and T is distinct from M, $MT = TN$ need not imply $M^*T = TM^*$; wherein both M and T are isometries. This suggests that the real interest remains about three distinct operators M, N and T with M and N quasinormal operators.

As an additional motivation a conditional validity of FPR-theorem proved by Brown and Douglas (1965) also indicates that the problem is worth considering. They show that; "Let V be an isometry on H, and T in $B(H)$ commutes with V. If the subspace $(VH)^{\perp}$ is invariant under T, then T commutes with V^*". Our considerations incidently lead us to the conclusion that the condition of the invariance of $(VH)^{\perp}$ under T cannot be dropped from this result.

In order to supply examples to falsify the FPR-theorem we divide the problem into two distinct possibilities.

Example 1 — Let $(x_j)_0^{\infty}$ be an orthonormal basis in l_2^+. Define operators M, N and T on l_2^+ as follows :

$$Mx_0 = x_1, \;\; Mx_1 = e^{i\theta}x_2, \;\; Mx_2 = x_3$$

and

$$Mx_j = \lambda_j x_{j+1} \;\text{ for } j \geqslant 3 \;\text{ and } (\lambda_j)_3^{\infty}$$

is a sequence of complex numbers of unit moduli.

$$Tx_0 = ix_1, \;\; Tx_1 = e^{i\theta}x_2, \;\; Tx_2 = e^{i\theta}x_3$$

and

$$Tx_j = \lambda_j x_{j+1} \;\text{ for } j \geqslant 3.$$

$$Nx_0 = ix_1, \;\; Nx_1 = x_2, \;\; Nx_2 = e^{i\theta}x_3$$

and

$$Nx_j = \lambda_j x_{j+1} \text{ for } j \geqslant 3.$$

Throughout θ is some fixed real number. One notes that $M \neq N \neq T$. The operators M, N are the weighted shifts with weights as complex numbers of unit moduli; and hence are isometries. One now checks easily that $MT = TN$, however, $M^*T \neq TN^*$.

Example 2 — As in Example 1 let $(x_j)_0^\infty$ be an orthonormal basis in l_2^+. Define operators T and M on l_2^+ as follows :

$Tx_j = \lambda x_{j+1}$ and $Mx_j = \mu x_{j+1}$ where $\lambda \neq \mu$ are complex numbers. Clearly T and M are quasinormal operators that are non-isometric and also $MT = TM$ holds. However, $M^*Tx_0 = \lambda M^*x_1 = \lambda\bar{\mu}x_0$ and $TM^*x_0 = 0$ showing that $M^*T \neq TM^*$.

The following example shows that the condition in the theorem of Brown and Douglas (1965) is necessary.

Example 3 — Consider $(x_n)_0^\infty$ to be an orthonormal basis in l_2^+ and define operators V by $Vx_n = x_{n+1}$ and $Tx_n = \lambda x_{n+1}$ for a complex scalar $\lambda \neq 0, 1$. Clearly V is an isometry and T is quasinormal. Also $TV = VT$ holds. But $(VH)^\perp = [x_0]$, the span of x_0. Hence for any complex μ_i, $\mu_i x_0 \in [x_0]$. But $T(\mu_i x_0) = \mu_i T(x_0) = \mu_i \lambda x_1 \notin [x_0]$ showing that x_0 is not invariant under T. Also $V^*T \neq TV^*$ on $[x_0]$.

Now we shall end this section with the following comments. It is known that the FPR-theorem can be used to give a very short proof of the fact that two similar normal operators are unitarily equivalent. Hoover (1972) has also shown that two similar isometries are unitarily equivalent; and we showed that FPR-Theorem fails for them. It is also known that two similar subnormal operators need not be unitarily equivalent. Hence, it will be worthwhile to investigate :

Open Problem — Whether two similar non-isometric quasinormal operators are unitarily equivalent?

Hoover (1972) solved this problem for isometries by using their Wold decomposition. Since a quasinormal operator is normal on its range, it is remarked that if a quasinormal operator can be expressed as the direct sum of a normal operator and a unilateral shift the problem may be solved in the affirmative. However a construction of non-normal quasinormal operators which are different from the scalar multiples of an isometry may perhaps shed some light on the solution of the problem in the negative direction.

3. Asymptotic-Type Properties

Motivated with FPR-theorem, Moore (1975) has discussed the problem : Is $\| N^*T - TN^* \|$ small whenever $\| NT - TN \|$ is small, where N is a normal operator and T is any operator on H?

Though a negative answer is supplied by Johnson and Williams (1975); operators they constructed are not uniformly bounded. Hence by controlling norm of T, Moore (1975) could supply an affirmative answer in the form of the following :

Result 3.1 — If K and ϵ are greater than 0, then there exists a $\delta > 0$ such that $\| T \| \leqslant K$ and $\| NT - TN \| < \delta$ implies $\| N^*T - TN^* \| < \epsilon$; where N is a normal operator.

Moore describes this behaviour as "Asymptotic Fuglede Theorem". A similar analogue of Result 3.1 can be expected for quasinormal operators due to a well-known characterization of such operators involving polar decomposition. The said analogue runs as follows :

Problem — Let $T = UP$ be the polar decomposition of a quasinormal operator. If $\epsilon > 0$, $c > 0$ are given, then there exists $\delta > 0$ such that for some S in $B(H)$, $\| S \| < c$ and $\| PS - SP \| \leqslant \delta$ imply $\| T^*S - ST^* \| \leqslant \epsilon$.

We, hereby produce an example to disprove this problem.

Counter Example — Let $(x_j)_0^\infty$ be an orthonormal basis in l_2^+ and define operators T and S by

$$Tx_j = 2x_{j+1} \text{ for all } j \geqslant 0;$$

and

$$Sx_0 = 2x_1, \text{ and } Sx_j = x_{j+1} \text{ for all } j = 1, 2, \ldots .$$

Clearly $\| S \| = 2$, and T is a quasinormal operator. In fact $T^*T = 4I$ and so $(T^*T)^{1/2} = P = 2I$. Hence $\| PS - SP \| = 0$ whereas $\| T^*S - ST^* \| = 4$.

So letting $\epsilon < 4$, we see that for no value of δ and $\| PS - SP \| < \delta$; we shall get $\| T^*S - ST^* \| < \epsilon$.

We now shift to the concepts of "near normal" and "almost normal" operators discussed by Bastian and Harrison (1974).

Definition — An operator T is called *near normal* if there exists $\delta > 0$ and a normal operator N satisfying $\| T - N \| \leqslant \delta$.

Following Bastian and Harrison (1974) we prove a relation between almost unitary and near unitary operators; the definitions of these notions should be obvious.

Result 3.2 — If H is a finite-dimensional Hilbert space and if $\epsilon > 0$, then there exists a $\delta > 0$ such that $\| I - A^*A \| \leqslant \delta$ for a contraction A implies that there exists a unitary operator U satisfying $\| U - A \| \leqslant \epsilon$.

PROOF : Let S be a unit boll in $B(H)$. For each $A \in S$ define a function f by $f(A) = \| I - A^*A \|$. Then clearly, f is real-valued continuous function on a compact metric space. Next define a set $N = \{A \in S : f(A) = 0\}$. Clearly, N is the set of unitary operators in S. If

$$M = \{A \in S : \| A - U \| > \epsilon \text{ for all } U \text{ in } N\}$$

then M is a compact subset of S on which f does not vanish. Define

$$\delta = \inf\{f(A) : A \in M\}.$$

One notes that $\delta > 0$; further if $f(A) < \delta$ for A in S, then A is not in M and we are through.

Note : In the above proof, definition of $f(A)$ and finiteness of H together imply that normality of A is not necessary.

Thus we see that on a finite-dimensional Hilbert space, 'almost unitary' operators are 'near unitary'.

An operator T is called *quasi-unitary* (or quasi-isometry) if

$$T^*T = TT^* = T + T^* \text{ (or } T^*T = T + T^*).$$

In short, we use the terms QU-operators and QI-operators respectively. As a consequence of our study above one observes that on a finite-dimensional Hilbert space QI-operator is a QU-operator. For details see Khasbardar (1977), and also Phadke *et al.* (1976). Using the technique of Result 3.2 one can arrive at the following :

Result 3.3 — If H is a finite-dimensional Hilbert space and if $\epsilon > 0$ is given, then there exists $\delta > 0$ such that $\| A + A^* - A^*A \| \leqslant \delta$ for a contraction A implies that there is a QU-operator Q in $B(H)$ satisfying $\| Q - A \| \leqslant \epsilon$.

We end this section by showing that Result 3.2 fails for infinite-dimensional Hilbert space when the operators are "almost isometries".

Example — Let U be the unilateral shift on l_2^+; and hence U is an isometry, $\| U \| = 1$. Put $A = U$ and $0 < \epsilon < 2$ in Result 3.2. We see that

$$\| I - A^*A \| = \| I - U^*U \| = 0 < \delta$$

for every non-zero δ. Whereas $\| U - V \| = 2 > \epsilon$ for every unitary operator (Halmos 1967), and this serves our purpose.

It is remarked that since the distance of the set of normal operators is 1 from the unilateral shift (*see* Halmos 1967) on infinite dimensional Hilbert spaces 'almost isometry' is not even 'near normal'.

4. Additional Properties and New Concepts

In this section we shall concentrate on some interesting properties of quasinormal operators that have not found any opening so far. In view of the consideration of Rudin (1974) for normal operators we can obtain, by mere computation, the following characterization.

"Let N and T be two commuting operators with T invertible. Then N is quasinormal if and only if TNT^{-1} is so". Further, it could also be shown that, "If N commutes with T^*T, then N is quasinormal if and only if TNT^* (or T^*NT) is so."

It can be observed that every power of a quasinormal operator is again quasinormal. Moreover, for a quasinormal T, the relation $(T^*T)^n = T^{*n}T^n$ holds for every positive integer n. This could be used to prove,

Lemma 4.1 — Every power of a quasinormal expansion is again an expansion (i.e. $I \leqslant T^*T$).

This lemma has its use in the proof of the following result which demonstrates the existence of an invariant subspace for a quasinormal expansion.

Result 4.2 — Let T be a quasinormal expansion on H. Then there exists an invariant subspace M of T and T/M is an isometry.

Proof : Define $M = \{X \in H : \| T^n x \| = \| x \|\}$; $(n = 0, 1, 2, \ldots)$. As a consequence of Lemma 4.1, T^n is an expansion for each n. Thus M is the subspace formed by the vectors x for which $(T^{*n}T^n - I)\, x = 0$. This shows that M is closed and $\| T^n Tx \| = \| T^{n+1}x \| = \| x \| = \| Tx \|$ for $x \in M$ $(n = 0, 1, 2, \ldots)$ means that M is invariant under T. That T/M is an isometry follows from the definition of M itself.

Let V be an isometry; then it is known that $M = H \ominus VH$ is a wandering subspace of V; *see* Sz-Nagy and Foias (1970) for definitions and notations. We shall generalize this in the following :

Result 4.3 — If T is a quasinormal operator then $\mathfrak{N}(T^*)$ is a wandering subspace for T.

Note that for any operator T, $\mathfrak{N}(T^*) = R(T)^{\perp}$ and hence M given above is $\mathfrak{N}(T^*)$.

Proof : It will suffice to show that $(T^p x, T^q x) = 0$ for all x in $\mathfrak{N}(T)$ with p, q positive integers and $p \neq q$. Of course, x in $\mathfrak{N}(T^*)$ gives $T^*x = 0$. If $p > q$ then

$$(T^p x, T^q x) = (T^{*q} T^q T^{p-q} x, x) = ((T^*T)^q T^{p-q} x, x)$$
$$= (T(T^*T)^q T^{p-q-1} x, x) = 0.$$

This proves the assertion.

To this end we introduce and discuss some operators which are very close to being quasinormals.

Definitions — An operator T is called

(a) Skew-quasinormal if $TT^*T = - T^*T^2$

(b) Co-quasinormal if $T = UP$ with U a co-isometry and $P = (TT^*)^{1/2}$ and P commutes with U.

It is known that T is called binormal if T^*T commutes with TT^*.

Result 4.4 — Every skew-quasinormal operator is binormal.

PROOF : Since $TT^*T = - T^*T^2$, $T^*TT^* = - T^{*2}T$ and thus

$$(TT^*)(T^*T) = - TT^*TT^* = (T^*T)(TT^*).$$

From the definition one can easily check that if T is co-quasinormal then T commutes with TT^*. This concept is introduced to find an analogue of the following result which we first prove.

Result 4.5 — If T is a quasinormal operator then there exists a coisometry V satisfying $T^* = VT$.

PROOF : Since T is quasinormal, $T = UP$ with U an isometry and P a positive operator; moreover P and U commute. Clearly $T^*U^*P = U^*U^*UP = U^{*2}T$. Now putting $V = U^{*2}$ we are through.

Exactly on the same lines one can show

Result 4.6 — If T is a co-quasinormal operator then there exists an isometry U satisfying $T^* = UT$.

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