REMARKS ON SOME FIXED POINT THEOREMS IN BIMETRIC SPACES

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Some common fixed point theorems for a family of mappings defined on a Bimetric space i.e. a triple (X, d_1, d_2) where d_1 and d_2 are metrics on X, are proved.

1. INTRODUCTION

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction mapping if

$$d(Tx, Ty) \leqslant \alpha d(x, y)$$

where $x, y \in X$ and $\alpha \in [0, 1)$. The well-known Banach contraction principle states that every contraction mapping on a complete metric space has a unique fixed point. Generalizing the above result, Maia (1968) proved the following.

1.1. *Theorem A* — Let $(X, d_1\, d_2)$ be a Bimetric space such that the following conditions hold :

(1.1.1) $d_1(x, y) \leqslant d_2(x, y)$ for all $x, y \in X$;

(1.1.2) (X, d_1) is complete;

(1.1.3) $T : X \to X$ is continuous with respect to d_1;

(1.1.4) $T : X \to X$ is contraction with respect to d_2.

Then T has a unique fixed point.

Recently Iseki (1975a), (1975b) obtained the following generalizations of theorem A.

1.2. *Theorem B* — Let (X, d_1, d_2) be a Bimetric space such that the following conditions hold.

(1.2.1) $d_1(x, y) \leqslant d_2(x, y)$ for all $x, y \in X$;

(1.2.2) (X, d_1) is complete;

(1.2.3) two mappings $T_1, T_2 : X \to X$ are continuous with respect to d_1 and

$$d_2(T_1x, T_2y) \leqslant \alpha \{d_2(x, T_1x) + d_2(y, T_2y)\} + \beta\{d_2(x, T_2y) + d_2(y, T_1x)\} + \gamma\, d_2(x, y)$$

where $x, y \in X$, $\alpha, \beta, \gamma \geqslant 0$ and $2\alpha + 2\beta + \gamma < 1$. Then T_1 and T_2 have a unique common fixed point.

1.3. *Theorem C* — Let (X, d_1, d_2) be a Bimetric space and let $H = \{T_i : i \in I$, the set of positive integers$\}$ be a family of mappings on X which are continuous with respect to d_1 and the following conditions hold :

(1.3.1) $d_1(x, y) \leqslant d_2(x, y)$ for all $x, y \in X$;

(1.3.2) (X, d_1) is complete;

(1.3.3) there exists some $T_i \in H$ such that for each $T_j \in H$ $(i \neq j)$, there are positive integers m_i and m_j with

$$d_2(T_i^{m_i}(x), T_j^{m_j}(y)) \leqslant \alpha \{d_2(x, T_i^{m_i}(x)) + d_2(y, T_j^{m_j}(y))\}$$
$$+ \beta \{d_2(x, T_j^{m_j}(y)) + d_2(y, T_i^{m_i}(x))\}$$
$$+ \gamma d_2(x, y)$$

where $x, y \in X$, $\alpha, \beta, \gamma \geqslant 0$ and $2\alpha + 2\beta + \gamma < 1$.

Then H has a unique common fixed point.

2. Results

Our purpose in this section is to get the same results under considerably weaker conditions. We drop the completeness of the space (X, d_1) and improve the conditions (1.2.3) and (1.3.3) respectively. The continuity of the mappings considered is also not needed; continuity at a point will serve the purpose.

2.1 *Theorem* — Let (X, d_1, d_2) be a Bimetric space and the following conditions hold.

(2.1.1) $d_1(x, y) \leqslant d_2(x, y)$ for all $x, y \in X$;

(2.1.2) $T_1, T_2 : X \to X$ are such that

$$d_2(T_1x, T_2y) \leqslant q \max \{d_2(x, y), d_2(x, T_1x), d_2(y, T_2y), \tfrac{1}{2} [d_2(x, T_2y) + d_2(y, T_1x)]\}$$

where $x, y \in X$, $q \in [0, 1)$;

(2.1.3) there exists a point $x_0 \in X$ such that the sequence of iterates defined as

(*) $x_1 = T_1(x_0)$, $x_2 = T_2(x_1)$, ..., $x_{2n-1} = T_1(x_{2n-2})$, $x_{2n} = T_2(x_{2n-1})$,

has a subsequence $\{x_{n_k}\}$ converging to a point $y_0 \in (X, d_1)$;

(2.1.4) $T_1, T_2 : X \to X$ are continuous at the point y_0 with respect to d_1.

Then T_1 and T_2 have y_0 as the unique common fixed point.

PROOF : By condition (2.1.2) and a routine calculation it can be shown that the sequence $\{x_n\}$ defined in (*) is a Cauchy sequence in (X, d_2) (cf. Ciric 1974). Now, by condition (2.1.1) it follows that $\{x_n\}$ is a Cauchy sequence in (X, d_1). Since $\{x_n\}$ is a Cauchy sequence having a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to y_0 \in (X, d_1)$, it follows that $x_n \to y_0$. Now we shall show that y_0 is a common fixed point of T_1 and T_2 i.e. $T_1(y_0) = y_0 = T_2(y_0)$. As a consequence of the continuity of T_1 at y_0 we have

$$T_1(y_0) = T_1 \lim_{n\to\infty} x_{2n-2} = \lim_{n\to\infty} T_1(x_{2n-2}) = \lim_{n\to\infty} x_{2n-1} = y_0.$$

Similarly, $T_2(y_0) = y_0$.

To prove the uniqueness, let $z_0 \in X$ be such that $z_0 \neq y_0$ and $T_1(z_0) = T_2(z_0) = z_0$. Then,

$$d_2(y_0, z_0) = d_2(T_1 y_0, T_2 z_0).$$

Using condition (2.1.2), we have

$$\begin{aligned} d_2(y_0, z_0) &\leqslant q \max \{d_2(y_0, z_0), d_2(y_0, T_1 y_0), d_2(z_0, T_2 z_0), \tfrac{1}{2} [d_2(y_0, T_2 z_0) \\ &\qquad + d_2(z_0, T_1 y_0)]\} \\ &\leqslant q d_2(y_0, z_0) \end{aligned}$$

a contradiction to the supposition that $q \in [0, 1)$. Therefore $y_0 = z_0$. This completes the proof.

2.2. *Theorem* — Let (X, d_1, d_2) be a Bimetric space and, let $H = \{ T_i : i \in I$, the set of positive integers$\}$ be a family of mappings on X such that the following conditions hold.

(2.2.1) $d_1(x, y) \leqslant d_2(x, y)$ for all $x, y \in X$;

(2.2.2) there exists some $T_i \in H$ such that for each $T_j \in H (i \neq j)$, there are positive integers m_i and m_j with

$$d_2(T_i^{m_i}(x), T_j^{m_j}(y)) \leqslant q \max \{d_2(x, y), d_2(x, T_i^{m_i}(x)), d_2(y, T_j^{m_j}(y)),$$
$$\tfrac{1}{2} [d_2(x, T_j^{m_j}(y)) + d_2(y, T_i^{m_i}(x))]\}$$

where $x, y \in X$, $q \in [0, 1)$;

(2.2.3) there exists a point $x_0 \in X$ such that the sequence of interates defined as

$$(^{**})\ x_1 = T_i^{m_i}(x_0),\ x_2 = T_j^{m_j}(x_1),\ \ldots,\ x_{2n-1} = T_i^{m_i}(x_{2n-2}),$$
$$x_{2n} = T_j^{m_j}(x_{2n-1}),$$

has a subsequence $\{x_{n_k}\}$ converging to a point $y_0 \in (X, d_1)$;

(2.2.4) $T_i \in H$ is continuous with respect to d_1 at the point y_0 for each $i \in I$.

Then y_0 is the unique common fixed point of H.

PROOF: By condition (2.2.2) and a routine calculation it can be shown that the sequence $\{x_n\}$ defined in (**) is a Cauchy sequence in (X, d_2). Then by condition (2.2.1), it follows that $\{x_n\}$ is a Cauchy sequence in (X, d_1). Now, the convergence of $\{x_{n_k}\}$ to $y_0 \in (X, d_1)$ ensures the convergence of $\{x_n\}$ to y_0. Since T_i is continuous at the point y_0 for each $i \in I$, we have

$$T_i^{m_i}(y_0) = T_i^{m_i} \lim_{n\to\infty} x_{2n-2} = \lim_{n\to\infty} T_i^{m_i}(x_{2n-2}) = \lim_{n\to\infty} x_{2n-1} = y_0.$$

Similarly, $T_j^{m_j}(y_0) = y_0$. Hence y_0 is a common fixed point of $T_i^{m_i}$ and $T_j^{m_j}$. As in theorem (2.1), it can be shown that y_0 is the unique common fixed point of $T_i^{m_i}$ and $T_j^{m_j}$. Also a fixed point of $T_i^{m_i}$ or $T_j^{m_j}$ is a fixed point of T_i or T_j respectively, it follows that y_0 is the unique common fixed point of H. This completes the proof.

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