FINITE DIFFERENCE INEQUALITIES AND DISCRETE TIME CONTROL SYSTEMS

by B. G. Pachpatte, *Department of Mathematics and Statistics, Marathwada University, Aurangabad* 431004 (*Maharashtra*)

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The present paper deals with two fundamental finite difference inequalities and their applications to the stability problems of discrete time control system of the more general type.

1. Introduction

One of the most fundamental contributions to the mathematical theory of systems over the past fifteen years has been the formulation and characterization of the property of controllability, which was first done by Kalman (1960) for linear, time-invariant, discrete-time systems. Recently, numerous studies on the subject have appeared for discrete time control systems in several directions, *see*, Freeman (1965), Kalman and Bertram (1960), Lin and Varaiya (1967), Michel and Wu (1969), Pachpatte (1972) and Weiss (1972), to mention a few. Almost invariably the "Lyapunov technique" has been employed in one form or another in the development of conditions for stability and asymptotic stability of the discrete time control systems. However, in the late 1950's Lyapunov's method appeared to have been exhausted, and at about that time V. M. Popov introduced the frequency method, which is currently being used in differential control systems. In this paper we develop a theory which deals with the stability problems of a class of discrete time control systems of the more general type. To accomplish this we first establish two fundamental finite difference inequalities which provide us a powerful technique to study the desired discrete time control systems.

2. Finite Difference Inequalities

In this section we establish two fundamental finite difference inequalities which can be used as handy tools in the study of discrete time control systems of the more general type. Throughout this paper we make use of the notations and preliminaries referring to the author's paper (Pachpatte 1973).

Theorem 1 — Let $x(n)$, $p(n)$, $h(n)$, and $q(n)$ be real-valued nonnegative functions defined on N, for which the inequality

Dedicated to the sacred memory of my late beloved mother.

$$x(n) \leqslant x_0 + \sum_{s=n_0}^{n-1} p(s)\,[x(s) + h(s)] + \sum_{s=n_0}^{n-1} p(s)\left(\sum_{\tau=n_0}^{s-1} q(\tau)\,x(\tau)\right) \quad ...(1)$$

holds for all $n \in N$, where x_0 is a nonnegative constant. Then

$$x(n) \leqslant x_0 + \sum_{s=n_0}^{n-1} p(s)\left[h(s) + x_0 \prod_{t=n_0}^{s-1} (1 + p(t) + q(t)) + \sum_{t=n_0}^{s-1} p(t)\,h(t) \prod_{\tau=t+1}^{s-1} (1 + p(\tau) + q(\tau))\right] \quad ...(2)$$

for all $n \in N$.

PROOF : Define a function $m(n)$ by the right member of (1). Then

$$\triangle\, m(n) = p(n)\,[x(n) + h(n)] + p(n) \sum_{\tau=n_0}^{n-1} q(\tau)\,x(\tau),\ m(n_0) = x_0$$

which in view of (1) implies

$$\triangle\, m(n) \leqslant p(n)\,h(n) + p(n)\,[m(n) + \sum_{\tau=n_0}^{n-1} q(\tau)\,m(\tau)]. \quad ...(3)$$

If we put

$$v(n) = m(n) + \sum_{\tau=n_0}^{n-1} q(\tau)\,m(\tau),\ v(n_0) = m(n_0) = x_0 \quad ...(4)$$

then it follows from (3), (4) and the fact that $m(n) \leqslant v(n)$, that the inequality

$$v(n+1) - [1 + p(n) + q(n)]\,v(n) \leqslant p(n)\,h(n) \quad ...(5)$$

is satisfied for all $n \in N$. Multiplying by $\prod_{s=n_0}^{n} (1 + p(s) + q(s))^{-1}$ to both sides of the inequality (5) and summing up from n_0 to $n-1$, we obtain the estimate for $v(n)$ such that

$$v(n) \leqslant x_0 \prod_{s=n_0}^{n-1} (1 + p(s) + q(s)) + \sum_{s=n_0}^{n-1} p(s)\,h(s) \prod_{\tau=s+1}^{n-1} (1 + p(\tau) + q(\tau)).$$

Substituting this value of $v(n)$ in (3), we have

$$\triangle\, m(n) \leqslant p(n)\left[h(n) + x_0 \prod_{s=n_0}^{n-1} (1 + p(s) + q(s)) + \sum_{s=n_0}^{n-1} p(s)\,h(s) \prod_{\tau=s+1}^{n-1} (1 + p(\tau) + q(\tau))\right]$$

which implies the estimation for $m(n)$ such that

$$m(n) \leqslant x_0 + \sum_{s=n_0}^{n-1} p(s) \Big[h(s) + x_0 \prod_{t=n_0}^{s-1} (1 + p(t) + q(t)) + \Big(\sum_{t=n_0}^{s-1} p(t)\, h(t) \prod_{\tau=t+1}^{s-1} (1 + p(\tau) + q(\tau)) \Big) \Big].$$

Now, substituting this value of $m(n)$ in (1) we obtain the desired bound in (2).

Special form of Theorem 1 when $h(n) = 0$ is already established and profitably employed by this author in the analysis of finite difference equations in an earlier paper (Pachpatte 1973).

We next establish the following finite difference inequality, which basically involves the comparison principle and is useful in our subsequent discussion.

Theorem 2 — Let $a(n, s) \geqslant 0$ be defined for $n, s \in N$ and $W(n, x, y)$ be a nonnegative function defined for $n \in N$, $0 \leqslant x < \infty$, $0 \leqslant y < \infty$, and monotone increasing with respect to x and y for any fixed $n \in N$, and

$$m(n) \leqslant h(n) + \sum_{s=n_0}^{n-1} a(n, s)\, W(s, m(s), \phi(s)),$$

where m, h are defined for $n \in N$. Suppose that $r_h(n)$ is the solution of the equation

$$r(n) = h(n) + \sum_{s=n_0}^{n-1} a(n, s)\, W(s, r(s), r(s)), \qquad \ldots(6)$$

existing on N such that $m_h(n_0) \leqslant r(n_0)$, then there holds an inequality

$$m(n) \leqslant r_h(n),\ n \in N$$

provided

$$\phi(n) \leqslant r_h(n),\ n \in N.$$

PROOF : We assume that the inequality $m(n) \leqslant r_h(n)$ is not satisfied for all $n \in N$. Then there exists a $\beta > n_0$ in N such that $m(n) \leqslant r_h(n)$ for $n = n_0$, $n_0 + 1, \ldots, \beta - 1$, but $m(\beta) > r_h(\beta)$, i.e. $r_h(\beta) - m(\beta) < 0$. On the other hand

$$r_h(\beta) - m(\beta) \geqslant \sum_{s=n_0}^{\beta-1} a(\beta, s)\, [W(s, r_h(s), r_h(s)) - W(s, m(s), r_h(s))] \geqslant 0.$$

We have thus obtained a contradiction, and the desired result follows.

We note that the finite difference inequality established in Theorem 2 is a further generalization of Theorem 3 recently obtained by this author (Pachpatte 1973).

3. Discrete Time Control Systems

The direct or second method of Lyapunov has greatly advanced the study of stability of force-free differential or discrete time systems. Recently, Lyapunov's direct method has been successfully extended to the study of the bounded-input bounded-output (BIBO) stability of arbitrary, nonlinear, time-varying, discrete control systems by Lin and Varaiya (1967). In this section we are concerned with the stability and asymptotic stability behaviour of the solutions of discrete time control systems of the form

$$x(n+1) = A(n)\, x(n) + f(n, x(n), \sigma(n)),\ x(n_0) = x_0 \qquad \ldots(7)$$

with

$$\sigma(n) = u(n) + \sum_{s=n_0}^{s-1} k(n, s, x(s)) \qquad \ldots(8)$$

as a perturbation of the linear system

$$y(n+1) = A(n)\, y(n),\ y(n_0) = x_0. \qquad \ldots(9)$$

Here x, y, σ, u, f and k are the elements of R^r, the r-dimensional vector space $A(n)$ is a $r \times r$ matrix with det $A(n) \neq 0$, the functions f and k are defined on $N \times R^r \times R^r \to R^r$ and $N \times N \times R^r \to R^r$ respectively. The symbol $|\cdot|$ will denote some convent norm on R^r as well as a corresponding consistent matrix norm. We denote by $x_\sigma(n, n_0, x_0)$ the solution of (7) - (8) with $x_\sigma(n_0, n_0, x_0) = x_0$. Let $y(n, n_0, x_0)$ be the solution of (9) and let $Y(n)$ denote the fundamental solution matrix of the system (9) such that $Y(n_0) = I$ (the identity matrix).

Of the many types of stability which may be defined for dynamical systems, at least two are of special importance. These are bounded-input bounded-output (BIBO) stability and exponential stability. In this paper our interest lies in the following stability definitions.

Definition 1 — The system (7) - (8) is said to be BIBO 'stable' if for any solution $x_\sigma(n, n_0, x_0)$ of (7) - (8) with $|\sigma(n)| < c_1\delta (c_1 = \text{constant})$, for $\delta > 0$ and $n \in N$ such that $|x_0| < \delta$ implies

$$|x_\sigma(n, n_0, x_0)| < c\,\delta (c = \text{constant}), \text{ for } n \geqslant n_0.$$

If, further, $\lim_{n\to\infty} |x_\sigma(n, n_0, x_0)| = 0$, system (7) - (8) is said to be asymptotically BIBO stable.

Definition 2 — The system (7) - (8) is said to be 'exponentially asymptotically stable' if for any solution $x_\sigma(n, n_0, x_0)$ of (7) - (8) such that $|x_0| < \delta$ implies

$$|x_\sigma(n, n_0, x_0)| < c\delta e^{-\alpha(n-n_0)},$$

for $c \geqslant 0$, $\delta > 0$, $\alpha > 0$, and $n \geqslant n_0$.

Definition 3 — The system (7) - (8) is said to be 'uniformly slowly growing' if for any solution $x_\sigma(n, n_0, x_0)$ of (7) - (8) such that $|x_0| < \delta$ implies

$$| x_\sigma(n, n_0, x_0) | < c\delta e^{\alpha(n-n_0)},$$

for $c \geqslant 0$, $\delta > 0$, $\alpha > 0$, and $n \geqslant n_0$.

Definition 4 — System (9) will be called 'stable' if for any solution $y(n, n_0, x_0)$ of (9) such that $| x_0 | < \delta$ implies

$$| y(n, n_0, x_0) | < c\delta \ (c = \text{constant}), \text{ for } \delta > 0, \text{ and } n \geqslant n_0.$$

If, further, $\lim_{n\to\infty} | y(n, n_0, x_0) | = 0$, system (9) is said to be 'asymptotically stable'.

Definition 5 — We shall take eqn. (6) to be stable if $h(n) < \delta$ implies $r_h(n) < c_1\delta$ (c_1 = constant), for $\delta > 0$ and $n \geqslant n_0$.

If further, $\lim_{n\to\infty} r_h(n) = 0$, then eqn. (6) is said to be 'asymptotically stable'.

Our first theorem investigates that the BIBO stability and asymptotic BIBO stability behaviour of solutions of (7) – (8) depends upon the stability and asymptotic stability behaviour of solutions of (9) and (6).

Theorem 3 — Assume that the fundamental solution matrix $Y(n)$ of (9) satisfies

$$| Y(n)\ Y^{-1}(s + 1) | \leqslant a(n, s),\ 0 \leqslant s \leqslant n < \infty \quad \text{...(10)}$$

where $a(n, s) \geqslant 0$ is a real valued function defined for $n, s \in N$. Let the function f in (7) satisfy an inequality

$$| f(n, x(n), \sigma(n)) | \leqslant W(n, | x(n) | , | \sigma(n) |), n \in N \quad \text{...(11)}$$

where $\sigma(n)$ is as given in (8) and W is the same function as defined in Theorem 2. Then BIBO stability (asymptotic BIBO stability) of system (7) – (8) follows from the stability (asymptotic stability) of system (9) and the stability (asymptotic stability) of the eqn. (6) with $h(n) = | y(n, n_0, x_0) |$ where $y(n, n_0, x_0)$ is any solution of (9).

PROOF : Using the variation of constants formula, any solution $x_\sigma(n, n_0, x_0)$ of (7) – (8) is represented by

$$x_\sigma(n, n_0, x_0) = y(n, n_0, x_0) + \sum_{s=n_0}^{n-1} Y(n)\ Y^{-1}(s + 1)\ f(s, x(s, n_0, x_0), \sigma(s)). \quad \text{...(12)}$$

Using (10) and (11) in (12) and applying Theorem 2 we see that the inequality

$$| x_\sigma(n, n_0, x_0) | \leqslant r_h(n),\ n \geqslant n_0 \quad \text{...(13)}$$

holds provided

$$| \sigma(n) | \leqslant r_h(n),\ n \geqslant n_0 \quad \text{...(14)}$$

where $r_h(n)$ is a solution of (6) with $h(n) = | y(n, n_0, x_0) |$.

Since the system (9) is stable, we have

$$| y(n, n_0, x_0) | < c\delta, \ n \geqslant n_0$$

whenever $| x_0 | < \delta$.

Further the eqn. (6) is stable so that we have $r_h(n) < c_1\delta$ for $h(n) < c\delta$. Thus,

$$| \sigma(n) | < c_1\delta$$

and

$$| x_\sigma(n, n_0, x_0) | < c_1\delta, \ n \geqslant n_0$$

whenever $| x_0 | < \delta$, i.e. system (7) - (8) is BIBO stable.

From the inequality (14) it follows that if instead of stability of (9) and (6) we have asymptotic stability, then the system (7) - (8) will be asymptotically BIBO stable. This completes the proof of the theorem.

Necessary and sufficient conditions for BIBO stability of a different form of (7) have been obtained using entirely different techniques by Lin and Varaiya (1967). Specifically, it is shown by these authors that a discrete time control system is BIBO stable if and only if there exist Lyapunov functions possessing certain properties. We note that our Theorem 3 yields sufficient conditions for BIBO stability and asymptotic BIBO stability without using a Lyapunov technique.

Our next theorem shows that under some suitable conditions on the functions involved in (7) - (8), any solution of (7) - (8) is exponentially asymptotically stable.

Theorem 4 — Assume that the fundamental solution matrix $Y(n)$ of (9) verifies

$$| Y(n)\, Y^{-1}(s) | \leqslant Me^{-\alpha(n-s)}, \ 0 \leqslant s \leqslant n < \infty \qquad \ldots(15)$$

where M and α are positive constants. Let the functions f and k in (7) - (8) satisfy

$$| f(n, x(n), \sigma(n)) | \leqslant p(n) \left(| x(n) | + | \sigma(n) | \right), n \in N \qquad \ldots(16)$$

$$| k(n, s, x(s)) | \leqslant e^{-\alpha n} q(s) \, | x(s) | , \ n, s \in N \qquad \ldots(17)$$

where $p(n)$ and $q(n)$ are real-valued functions defined for $n \in N$ and

$$\sum_{s=n_0}^{\infty} e^{\alpha(1-n_0)} p(s) \Bigg[| u(s) | \, e^{\alpha s} + M \, | x_0 | \, e^{\alpha n_0} \prod_{t=n_0}^{s-1} Q(t)$$

$$+ \sum_{t=n_0}^{s-1} Mp(t) e^{\alpha(t+1)} \, | u(t) | \prod_{\tau=t+1}^{s-1} Q(\tau) \Bigg] \leqslant B \qquad \ldots(18)$$

where $Q(n) = (1 + Me^{\alpha}\, p(n) + e^{-\alpha n}\, q(n))$, $u(n)$ is the function given in (8) and $B > 0$ is a constant. Then any solution $x_\sigma(n, n_0, x_0)$ of (7) - (8) with $x(n_0) = x_0$ is exponentially asymptotically stable.

PROOF : By using the variation of constants formula any solution $x_\sigma(n, n_0, x_0)$ of (7) - (8) is represented by

$$x_\sigma(n, n_0, x_0) = Y(n)\, Y^{-1}(n_0)\, x_0 + \sum_{s=n_0}^{n-1} Y(n)\, Y^{-1}(s+1) \times f(s, x_\sigma(s, n_0, x_0), \sigma(s)). \quad \ldots(19)$$

Using (15), (16), (19), we obtain

$$| x_\sigma(n, n_0, x_0) | \leqslant | x_0 |\, Me^{-\alpha(n-n_0)} + \sum_{s=n_0}^{n-1} Me^{-\alpha(n-s-1)} p(s) \times [\, | x_\sigma(s, n_0, x_0) | + | \sigma(s) |\,]. \quad \ldots(20)$$

Further, using (8), (17) in (20) we have

$$| x_\sigma(n, n_0, x_0) | \leqslant | x_0 |\, Me^{-\alpha(n-n_0)} + \sum_{s=n_0}^{n-1} Me^{-\alpha(n-1)} p(s) \times [\, | x_\sigma(s, n_0, x_0) |\, e^{\alpha s} + | u(s) |\, e^{\alpha s}] + \sum_{s=n_0}^{n-1} Me^{-\alpha(n-1)} p(s) \left(\sum_{\tau=n_0}^{s-1} q(\tau)\, e^{-\alpha\tau}\, | x_\sigma(\tau, n_0, x_0) |\, e^{\alpha\tau} \right).$$

Multiplying both sides of the above inequality by $e^{\alpha n}$, applying Theorem 1 with $x(n) = | x_\sigma(n, n_0, x_0) |\, e^{\alpha n}$, then multiplying by $e^{-\alpha n}$, we obtain

$$| x_\sigma(n, n_0, x_0) | \leqslant Me^{-\alpha(n-n_0)} \Big[\, | x_0 | + \sum_{s=n_0}^{n-1} p(s)\, e^{\alpha(1-n_0)} \Big\{ | u(s) |\, e^{\alpha s} + M(x)\, e^{\alpha n_0} \prod_{t=n_0}^{s-1} Q(t) + \sum_{t=n_0}^{s-1} Mp(t) e^{\alpha(t+1)}\, | u(t) | \prod_{\tau=t+1}^{s-1} Q(\tau) \Big\} \Big].$$

The above estimation in view of the assumption (18) implies

$$| x_\sigma(n, n_0, x_0) | \leqslant Me^{-\alpha(n-n_0)}(\delta + B)$$

i.e. $$| x_\sigma(n, n_0, x_0) | \leqslant \delta' Me^{-\alpha(n-n_0)}, \; n \geqslant n_0$$

whenever $| x_0 | < \delta$, where $\delta' = \delta + B$. This proves the exponential asymptotic stability of the system (7) – (8).

Theorem 5 below demonstrates that, under some suitable conditions on the functions involved in (7) – (8), any solution of (7) – (8) is slowly growing.

Theorem 5 — Assume that the fundamental solution matrix $Y(n)$ of (9) verifies

$$| Y(n)\, Y^{-1}(s) | \leqslant Me^{\alpha(n-s)}, \; 0 \leqslant s \leqslant n < \infty \quad \ldots(21)$$

where M and α are positive constants. Let the functions f and k in (7) – (8) satisfy

$$| f(n, x(n), \sigma(n)) | \leqslant p(n)\, (\, | x(n) | + | \sigma(n) |\,), \; n \in N \quad \ldots(22)$$

$$| k(n, s, x(s)) | \leqslant e^{\alpha n} q(s) \, | x(s) | \, , \; n, s \in N \qquad \ldots(23)$$

where $p(n)$ and $q(n)$ are real-valued functions defined for $n \in N$ and

$$\sum_{s=n_0}^{\infty} e^{-\alpha(1-n_0)} p(s) \Big[\, | u(s) | \, e^{-\alpha s} + M \, | x_0 | \, ^{-\alpha n_0} \prod_{t=n_0}^{s-1} R(t)$$

$$+ \sum_{t=n_0}^{s-1} M p(t) \, e^{-\alpha(t+1)} \, | u(t) | \prod_{\tau=t+1}^{s-1} R(\tau) \Big] \leqslant B \qquad \ldots(24)$$

where $R(n) = (1 + Me^{-\alpha} p(n) + e^{\alpha n} q(n)$, $u(n)$ is the function given in (8) and $B > 0$ is a constant. Then any solution $x_\sigma(n, n_0, x_0)$ of (7) - (8) with $x(n_0) = x_0$ is uniformly slowly growing.

The proof of this theorem follows by the similar argument as in the proof of Theorem 4 with suitable modifications, and hence we omit the details.

Finally, we note that the results obtained in Theorems 4 and 5 appear to be new. Although the arguments presented in the proofs are probably not new, but the author has been unable to locate similar results in the literature.

4. An Example

In this section we give an interesting example to illustrate Theorem 4.

Consider the nonlinear discrete time control system (7) - (8) as a perturbation of the linear discrete time system (9) with

$$A(n) = e^{-1} \begin{pmatrix} \cos(e^{n+1} - e^n) & \sin(e^{n+1} - e^n) \\ -\sin(e^{n+1} - e^n) & \cos(e^{n+1} - e^n) \end{pmatrix}$$

and the functions f and k in (7) - (8) satisfy the hypotheses (16), (17) and (18) of Theorem 4 with $\alpha = 1$. Then the fundamental solution matrix of (9) satisfies

$$Y(n) = e^{-n} \begin{pmatrix} \sin e^n & -\cos e^n \\ \cos e^n & \sin e^n \end{pmatrix} \qquad \ldots(25)$$

$$Y^{-1}(n) = e^{n} \begin{pmatrix} \sin e^n & \cos e^n \\ -\cos e^n & \sin e^n \end{pmatrix}. \qquad \ldots(26)$$

One may very easily verify the condition (15) of Theorem 4 with respect to (25) and (26), for $\alpha = 1$, and hence we have

$$| Y(n) \, Y^{-1}(s) | \leqslant Me^{-(n-s)}, \; 0 \leqslant s \leqslant n < \infty, \; n, s \in N \qquad \ldots(27)$$

where $M > 0$ is a constant.

It is known, the solution $x_\sigma(n, n_0, x_0)$ of (7) - (8) is represented by the eqn. (19). Using (19), (27), (16), (8), (17) and following the similar argument as in the proof of Theorem 4 we obtain

$$| x_\sigma(n, n_0, x_0) | < \delta' M e^{-(n-n_0)}, \; n \geqslant n_0$$

whenever $| x_0 | < \delta$, where $\delta' = \delta + B$, and the conclusion of Theorem 4 is true.

5. Conclusions

In this paper, a theory is developed which is concerned with the stability criteria for a class of discrete time control systems by using finite difference inequalities of the Gronwall-Bellman type. The class of the systems considered is general enough so as to include linear and nonlinear systems, time invariant and time-varying systems, unforced and forced systems. Various definitions of stability are considered and the corresponding theorems are stated and proved. These theorems yield sufficient conditions for various forms of stability.

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