ON GENERALIZED BERNSTEIN POLYNOMIALS

by ABDUL WAFI, ANWAR HABIB *and* HUZOOR H. KHAN, *Department of Mathematics, Aligarh Muslim University, Aligarh* 202001

The object of this paper is to improve the result of Voronowskaja (1932) for the modified Bernstein polynomials $P_n(f; x)$.

1. INTRODUCTION AND RESULTS

If $f(x)$ is a function defined on [0, 1], the Bernstein polynomial $B_n(f)$ of f is

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x)$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Voronowskaja (1932) has proved the following theorem for the polynomial $B_n(f, x)$:

Theorem A — Let $f(x)$ be bounded in [0, 1] and suppose that the second derivative $f''(x)$ exists at a certain point x of [0, 1], then

$$\lim_{n \to \infty} n[f(x) - B_n(x)] = -\frac{x(1-x)}{2} f''(x).$$

In particular, if $f''(x) \neq 0$, the difference $f(x) - B_n(x)$ is exactly of order n^{-1}.

A small modification of Bernstein polynomials due to Kantorovič (1930) makes it possible to approximate Lebesgue integrable functions in L_1 norm by the modified polynomials

$$P_n(f, x) = (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right) p_{n,k}(x).$$

In this paper we shall show that the theorem A can easily be extended by taking the modified Bernstein polynomials $P_n(f, x)$. Our theorem may be stated as follows :

Theorem B — Let $f(x)$ be bounded Lebesgue integrable function with its first derivatives in [0, 1] and suppose that the second derivative $f''(x)$ exists at a certain point x of [0, 1], then

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$$\lim_{n\to\infty} n[f(x) - P_n(x)] = -\tfrac{1}{8} f''(x). \qquad \text{...(1.1)}$$

In particular, if $f''(x) \neq 0$, the difference $f(x) - P_n(x)$ is exactly of order n^{-1}.

2. Proof of Theorem B

To prove (1.1) we write

$$f(t) = f(x) + (t - x) f'(x) + (t - x)^2 [\tfrac{1}{2} f''(x) + \eta(t - x)] \qquad \text{...(2.1)}$$

where $\eta(h)$ is bounded, $|\eta(h)| \leqslant H$ for all h and converges to zero with h.

Multiplying eqn. (2.1) by $(n + 1)\, p_{n,k}(x)$ and integrating it from $k/(n + 1)$ to $(k + 1)/(n + 1)$, then on summing we get

$$(n + 1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right) p_{n,k}(x)$$

$$= (n + 1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} f(x)\, dt \right) p_{n,k}(x)$$

$$+ (n + 1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t - x) f'(x)\, dt \right) p_{n,k}(x)$$

$$+ \tfrac{1}{2}(n + 1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t - x)^2 f''(x)\, dt \right) p_{n,k}(x)$$

$$+ (n + 1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t - x)^2 \eta(t - x)\, dt \right) p_{n,k}(x)$$

$$= f(x) + (n + 1) \sum_{k=0}^{n} \left(\left\{ \left[\frac{t^2}{2} \right]_{k/(n+1)}^{(k+1)/(n+1)} - x \left[t \right]_{k/(n+1)}^{(k+1)/(n+1)} \right\} \right.$$

$$\left. \times f'(x) \right) p_{n,k}(x)$$

$$+ \tfrac{1}{2}(n + 1) \sum_{k=0}^{n} \left(\left\{ \left[\frac{t^3}{3} \right]_{k/(n+1)}^{(k+1)/(n+1)} - x \left[t^2 \right]_{k/(n+1)}^{(k+1)/(n+1)} \right. \right.$$

$$\left. \left. + x^2 \left[t \right]_{k/(n+1)}^{(k+1)/(n+1)} \right\} f''(x) \right) p_{n,k}(x)$$

$$+ (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \eta(t-x)\,dt \right) p_{n,k}(x)$$

$$= f(x) + (n+1) \sum_{k=0}^{n} \left(\left[\frac{2k+1}{2(n+1)^2} - \frac{x}{(n+1)} \right] f'(x) \right) p_{n,k}(x)$$

$$+ \tfrac{1}{2}(n+1) \sum_{k=0}^{n} \left(\left[\frac{3k^2+3k+1}{3(n+1)^3} - \frac{2kx+x}{(n+1)^2} + \frac{x^2}{(n+1)} \right] \right.$$

$$\left. \times f''(x) \right) p_{n,k}(x)$$

$$+ (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \eta(t-x)\,dt \right) p_{n,k}(x)$$

$$= f(x) + \frac{1-2x}{2(n+1)} f'(x) + \tfrac{1}{2}(n+1) \left[\frac{3nx(1-x) - 3x(1-x) + 1}{3(n+1)^3} \right]$$

$$\times f''(x) + (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \eta(t-x)\,dt \right) p_{n,k}(x).$$

Since $x(1-x) \leqslant 1/4$ on $[0, 1]$, then we obtain the inequality

$$\leqslant f(x) + \left| \frac{1-2x}{2(n+1)} f'(x) \right| + \tfrac{1}{2}(n+1) \left[\frac{\frac{3n}{4} - \frac{3}{4} + 1}{3(n+1)^3} \right] f''(x)$$

$$+ (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x^2) \eta(t-x)\,dt \right) p_{n,k}(x)$$

$$\leqslant f(x) + \frac{1}{2(n+1)} \, | f'(x) | + \frac{1}{8n} f''(x)$$

$$+ (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \eta(t-x)\,dt \right) p_{n,k}(x)$$

$$\leqslant f(x) + \frac{M}{n} + \frac{1}{8n} f''(x)$$

$$+ (n+1) \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \eta(t-x)\,dt \right) p_{n,k}(x)$$

where $| f'(x) | \leqslant M$.

The last term on the right-hand side can easily be estimated. Let $\epsilon > 0$ be arbitrary and $\delta > 0$ such that $|h| < \delta$ implies $|\eta(h)| < \epsilon$, then its absolute value does not exceed

$$\epsilon(n+1) \sum_{|t-x| \leqslant \delta} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \, dt \right) p_{n,k}(x)$$

$$+ (n+1) H \sum_{|t-x| > \delta} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} dt \right) p_{n,k}(x),$$

$$\leqslant \frac{\epsilon}{4n} + (n+1) H\delta^{-2} \sum_{k=0}^{n} \left(\int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \, dt \right) p_{n,k}(x)$$

$$\leqslant \frac{\epsilon}{4n} + \frac{H\delta^{-2}}{4n} = \frac{\epsilon + H\delta^{-2}}{4n} = \frac{\epsilon_n'}{n}.$$

Thus

$$P_n(f, x) = f(x) + \frac{M}{n} + \frac{1}{8n} f''(x) + \frac{\epsilon_n'}{n}$$

$$= f(x) + \frac{1}{8n} f''(x) + \frac{M + \epsilon_n'}{n},$$

$$P_n(f, x) = f(x) + \frac{1}{8n} f''(x) + \frac{\epsilon_n}{n}. \qquad \ldots(2.2)$$

For $\epsilon_n \to 0$ and $n \to \infty$, (2.2) is equivalent to (1.1), i.e.

$$\lim_{n \to \infty} n[f(x) - P_n(f, x)] = -\tfrac{1}{8} f''(x).$$

This completes the proof of the Theorem B.

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