CONFORMALLY FLAT NON-STATIC SPHERICALLY SYMMETRIC PERFECT FLUID DISTRIBUTIONS IN GENERAL RELATIVITY

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Solutions of Einstein's field equations representing spherically symmetric perfect fluid distribution which are conformally flat, have been derived. Various physical properties of the models have also been discussed.

1. Introduction

Conformally flat space-times are of particular interest in view of their degeneracy in the context of Petrov classification. A number of physically significant metrics are conformally flat like Schwarzschild internal solution and Lemaitre universe. A conformally flat spherically symmetric non-static internal solution was obtained by Singh and Abdussattar (1974). In this paper, we obtain a general solution representing conformally flat perfect fluid distribution of spherical symmetry. The explicit expressions for pressure, density, expansion, rotation, shear and non-vanishing components of flow vector have also been obtained.

We consider the metric in the form

$$dS^2 = e^{\lambda}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - dt^2) \qquad \ldots(1)$$

where λ is a function of r and t alone. The energy momentum tensor for a perfect fluid distribution is given by

$$T_{ij} = (\epsilon + p)\, v_i v_j + p g_{ij} \qquad \ldots(2)$$

together with

$$g_{ij}\, v^i v^j = -1. \qquad \ldots(3)$$

The field equations

$$R_{ij} - \tfrac{1}{2} R g_{ij} + \Lambda g_{ij} = -8\pi T_{ij} \qquad \ldots(4)$$

for the line element (1) are

$$\frac{3\lambda_1^2}{4} + \frac{2\lambda_1}{r} - \lambda_{44} - \frac{\lambda_4^2}{4} - \Lambda e^{\lambda} = 8\pi\, [(\epsilon + p)\, v_1^2 + p e^{\lambda}] \qquad \ldots(5)$$

$$\lambda_{11} + \frac{\lambda_1^2}{4} + \frac{\lambda_1}{r} - \lambda_{44} - \frac{\lambda_4^2}{4} - \Lambda e^{\lambda} = 8\pi p e^{\lambda} \qquad ...(6)$$

$$-\lambda_{11} - \frac{\lambda_1^2}{4} - \frac{2\lambda_1}{r} + \frac{3\lambda_4^2}{4} + \Lambda e^{\lambda} = 8\pi\left[(\epsilon + p)\, v_4^2 - p e^{\lambda}\right] \qquad ...(7)$$

and

$$-\lambda_{14} + \frac{\lambda_1 \lambda_4}{2} = 8\pi(\epsilon + p)\, v_1 v_4. \qquad ...(8)$$

Equation (3) reduces to

$$v_4^2 - v_1^2 = e^{\lambda}. \qquad ...(9)$$

In the above the suffixes 1 and 4 denote differentiation with respect to r and t respectively.

2. Solution of the Field Equations

From eqns. (5) and (6), we have

$$8\pi(\epsilon + p)\, v_1^2 = \frac{\lambda_1^2}{2} + \frac{\lambda_1}{r} - \lambda_{11}. \qquad ...(10)$$

From eqns. (6) and (7), we have

$$8\pi(\epsilon + p)\, v_4^2 = \frac{\lambda_4^2}{2} - \frac{\lambda_1}{r} - \lambda_{44}. \qquad ...(11)$$

Equations (9), (10) and (11) lead to

$$8\pi(\epsilon + p)\, e^{\lambda} = \frac{\lambda_4^2}{2} - \lambda_{44} - \frac{\lambda_1^2}{2} + \lambda_{11} - \frac{2\lambda_1}{r}. \qquad ...(12)$$

From eqns. (6) and (12), we have

$$8\pi\epsilon = e^{-\lambda}\left[\frac{3\lambda_4^2}{4} - \frac{3\lambda_1^2}{4} - \frac{3\lambda_1}{r}\right] + \Lambda. \qquad ...(13)$$

From eqns. (10), (11) and (8), we have

$$\lambda_{11}\left(\frac{\lambda_1}{r} - \frac{\lambda_4^2}{2}\right) + \lambda_{14}\lambda_1\lambda_4 + \lambda_{44}\left(-\frac{\lambda_1^2}{2} - \frac{\lambda_1}{r}\right) + (\lambda_{11}\lambda_{44} - \lambda_{14}^2) = \frac{\lambda_1}{2r}\left(\lambda_1^2 - \lambda_4^2 + \frac{2\lambda_1}{r}\right). \qquad ...(14)$$

If we apply Monge's method in eqn. (14), we find following solution

$$e^{\lambda} = \frac{16m^2}{[\phi(mr^2 - mt^2 - t) - nt]^2} \qquad ...(15)$$

where m and n are constants. Hence the metric reduces to the form

$$dS^2 = \frac{16m^2}{(\phi - nt)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - dt^2). \qquad ...(16)$$

The solution obtained by Singh and Abdussattar (1974) is a particular case of (16) with $n = 0$. Also when $n = 0$, the metric (16) transforms to Robertson Walker metric of constant negative curvature.

3. Some Physical Properties

The pressure and density for the model (16) are given by

$$8\pi p = \frac{(\phi - nt)^2}{16m^2}\left[(1 - 4m\xi)\left(\frac{2\phi''}{\phi - nt} - \frac{3\phi'^2}{(\phi - nt)^2}\right) - \frac{12m\phi'}{\phi - nt} - \frac{6n(2mt + 1)\phi'}{(\phi - nt)^2} - \frac{3n^2}{(\phi - nt)^2}\right] - \Lambda \qquad ...(17)$$

and

$$8\pi\epsilon = \frac{(\phi - nt)^2}{16m^2}\left[\frac{3(1 - 4m\xi)\phi'^2}{(\phi - nt)^2} + \frac{12m\phi'}{\phi - nt} + \frac{6n(2mt + 1)\phi'}{(\phi - nt)^2} + \frac{3n^2}{(\phi - nt)^2}\right] + \Lambda \qquad ...(18)$$

where $\xi = mr^2 - mt^2 - t$ and a prime indicates differentiation with respect to its argument. The non-vanishing components of flow vector are given by

$$v_1 = \frac{8m^2 r}{(\phi - nt)\sqrt{1 - 4m\xi}} \qquad ...(19)$$

and

$$v_4 = -\frac{4m(2mt + 1)}{(\phi - nt)\sqrt{1 - 4m\xi}}. \qquad ...(20)$$

The model has to satisfy the reality condition (Ellis 1971)

(i) $(\epsilon + p) > 0$

and

(ii) $(\epsilon + 3p) > 0$.

Condition (i) leads to

$$(1 - 4m\xi)(\phi - nt)\phi'' > 0. \qquad ...(21)$$

Since $(1 - 4m\xi) > 0$, we have from eqn. (21), that $(\phi - nt)$ and ϕ'' should have the same sign. The condition (ii) leads to

$$(1 - 4m\xi)\left(\frac{\phi''}{\phi - nt} - \frac{\phi'^2}{(\phi - nt)^2}\right) - \frac{4m\phi'}{\phi - nt} - \frac{2n(2mt + 1)\,\phi'}{(\phi - nt)^2}$$

$$- \frac{n^2}{(\phi - nt)^2} > \frac{16\,\Lambda\, m^2}{3(\phi - nt)^2} \qquad \text{...(22)}$$

which imposes condition on Λ. The non-vanishing components of the vector $\dot{v}_i = v_{i;j}v^j$ are

$$\dot{v}_1 = \frac{2mrn(2mt + 1)}{(\phi - nt)\,(1 - 4m\xi)} \qquad \text{...(23)}$$

and

$$\dot{v}_4 = -\frac{n(2mt + 1)^2}{(\phi - nt)\,(1 - 4m\xi)}. \qquad \text{...(24)}$$

The flow is therefore non-geodetic in general.

The expressions for expansion θ', rotation ω_{ij} and shear σ_{ij} calculated for the flow vector v^i are given by

$$\theta' = \frac{3(\phi - nt)}{2\sqrt{1 - 4m\xi}} + \frac{3\phi'\sqrt{1 - 4m\xi}}{4m} - \frac{3n(2mt + 1)}{4m\sqrt{1 - 4m\xi}} \qquad \text{...(25)}$$

$$\omega_{14} = \frac{4m^2rn}{(\phi - nt)^2\sqrt{1 - 4m\xi}} \qquad \text{...(26)}$$

and

$$\sigma_{11} = \frac{8mn(2mt + 1)\,\{1 - 4m\xi + 4m^2r^2\}}{(\phi - nt)^2\,(1 - 4m\xi)^{3/2}} \qquad \text{...(27)}$$

$$\sigma_{22} = \frac{8mnr^2\,(2mt + 1)}{(\phi - nt)^2\sqrt{1 - 4m\xi}} \qquad \text{...(28)}$$

$$\sigma_{33} = \frac{8mnr^2\sin^2\theta(2mt + 1)}{(\phi - nt)^2\sqrt{1 - 4m\xi}} \qquad \text{...(29)}$$

$$\sigma_{44} = \frac{4mn(2mt + 1)\,\{2(2mt + 1)^2 - (1 - 4m\xi)\}}{(\phi - nt)^2\,(1 - 4m\xi)^{3/2}} \qquad \text{...(30)}$$

and

$$\sigma_{14} = -\frac{4m^2rn\,\{4(2mt + 1)^2 + 1 - 4m\xi\}}{(\phi - nt)^2\,(1 - 4m\xi)^{3/2}}. \qquad \text{...(31)}$$

The other components of rotation tensor ω_{ij} and shear tensor σ_{ij} vanish. Hence the model is expanding, rotating, shearing but non-geodetic in general. The metric (16) cannot represent a dust distribution or disordered radiation unless $n = 0$. Clearly the components $\dot{v}_1$, and $\dot{v}_4$ vanish when $n = 0$ in which case the metric (16) transforms to Robertson Walker metric.

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