LIMITING OPTIMAL EXPLOITATION OF A MULTISPECIES COMMUNITY

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(*Received* 3 *March* 1978)

Limiting optimal solutions for maximizing the present value of discounted profits for exploitation of a community of n species have been obtained. The second variation has also been considered to determine whether the optimal solution gives a local maximum.

1. THE OPTIMIZATION PROBLEM

We consider the problem of optimal exploitation of a community of n species of fish with natural rates of growth given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \text{ or } \frac{dx_i}{dt} = F_i(x_1, x_2, \ldots, x_n), (i = 1, 2, \ldots, n) \qquad \ldots(1)$$

where $x_1(t), x_2(t), \ldots, x_n(t)$ are the populations of the n species of fish at time t. Due to the effort $E(t)$ made at time t, the catch per unit time is assumed to be

$$\mathbf{G}(\mathbf{x})\, E + \mathbf{H}(\mathbf{x}), \qquad \ldots(2)$$

so that the rates of growth of exploited populations are given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) - E\,\mathbf{G}(\mathbf{x}) - \mathbf{H}(\mathbf{x}) \qquad \ldots(3)$$

Following Sancho and Mitchell (1976), we take the cost of making an effort E given by

$$q(E) = bE^2 + kE + l \qquad \ldots(4)$$

Let p_i be the selling price of the ith species and let λ correspond to the instantaneous rate of discount, then the present value of discounted profits is given by

$$\int_0^\infty e^{-\lambda t} [p_i G_i(\mathbf{x})\, E + p_i H_i(\mathbf{x}) - bE^2 - kE - l]\, dt \qquad \ldots(5)$$

where we are using the summation convention, i.e., whenever a suffix is repeated, summation is understood with respect to it.

We want to choose $E(t)$ so as to maximize the expression (5) subject to (3). For this purpose we use the maximum principle of Pontryagin (1963).

2. Use of Maximum Principle

The Hamiltonian is given by

$$H = e^{-\gamma t}\,[p_i G_i(\mathbf{x})\,E + p_i H_i(\mathbf{x}) - bE^2 - kE - l] + \psi_i(t)\,[F_i(\mathbf{x}) - EG_i(\mathbf{x}) - H_i(\mathbf{x})]. \quad \ldots(6)$$

Since H has to be a maximum as a function of E, we get

$$e^{-\lambda t}\,[p_i G_i(\mathbf{x}) - 2bE - k] - \psi_i(t)\,G_i(\mathbf{x}) = 0. \quad \ldots(7)$$

We also get the auxiliary equations

$$e^{-\lambda t}\left[p_i \frac{\partial G_i}{\partial x_j} E + p_i \frac{\partial H_i}{\partial x_j}\right] + \psi_i(t)\left[\frac{\partial F_i}{\partial x_j} - E\frac{\partial G_i}{\partial x_j} - \frac{\partial H_i}{\partial x_j}\right] = -\frac{d\psi_j}{dt},\ (j = 1, 2, \ldots, n). \quad \ldots(8)$$

Equations (3) give

$$\frac{dx_i}{dt} = F_i(\mathbf{x}) - EG_i(\mathbf{x}) - H_i(\mathbf{x}),\ (i, = 1, 2, \ldots, n). \quad \ldots(9)$$

Equations (7), (8), (9) give $2n$ differential and one algebraic equation to determine

$$x_1(t), x_2(t), \ldots, x_n(t);\ \psi_1(t), \psi_2(t), \ldots, \psi_n(t);\ E(t). \quad \ldots(10)$$

The solution of these equations for the case when

$$F_i(\mathbf{x}) = a_i x_i,\ \ G_i(\mathbf{x}) = \gamma_i,\ \ H_i(\mathbf{x}) = \alpha_i + \beta_{ij} x_j \quad \ldots(11)$$

has been discussed in Kapur and Saleem (1977). The summation convention is not used in the first equation.

For general functions $\mathbf{F}(\mathbf{x}), \mathbf{G}(\mathbf{x}), \mathbf{H}(\mathbf{x})$ it is not possible to solve eqns. (7), (8) and (9). Clark (1976) examined the limiting optimal solution, i.e., the solution in the case when $x_i(t)$ and $E(t)$ are constants. Kapur and Saleem (1978) discussed the two species case further and obtained the conditions for the existence of the solutions for some special cases. Silvert and Smith (1977) have discussed a closely related problem for which they have also examined the sign of the second variation.

In the present paper, we discuss the limiting optimal solutions and the sign of the second variation for the present problem.

3. Limiting Optimal Solutions

We consider the existence of solutions of the following type :

$$\mathbf{x} = \mathbf{x}_0,\ x_i = (x_i)_0,\ E = E_0,\ \psi_i(t) = C_i\, e^{-\lambda t}. \quad ...(12)$$

Substituting in (7), (8), (9), we get the equations

$$[p_i G_i(\mathbf{x}_0) - 2bE_0 - k] - C_i G_i(\mathbf{x}_0) = 0 \quad ...(13)$$

$$p_i\left(\frac{\partial G_i}{\partial x_j}\right)_0 E_0 + p_i\left(\frac{\partial H_i}{\partial x_j}\right)_0 + C_i\left(\frac{\partial F_i}{\partial x_j}\right)_0 - C_i E_0\left(\frac{\partial G_i}{\partial x_j}\right)_0$$

$$- C_i\left(\frac{\partial H_i}{\partial x_j}\right)_0 = \lambda C_j,\ (j = 1, 2, ..., n) \quad ...(14)$$

$$F_i\,(\mathbf{x}_0) - E_0 G_i(\mathbf{x}_0) - H_i(\mathbf{x}_0) = 0,\ \ (i = 1, 2, ..., n). \quad ...(15)$$

Equations (13), (14) and (15) give us $(2n + 1)$ algebraic equations to solve for

$$(x_1)_0,\ (x_2)_0,\ ...,\ (x_n)_0;\ E_0;\ C_1, C_2, ..., C_n. \quad ...(16)$$

For the solutions to be feasible, it is necessary that

$$\mathbf{x}_0 \geqslant 0,\ E_0 > 0. \quad ...(17)$$

If these conditions are satisfied, then the optimal discounted revenue is given by

$$\frac{1}{\lambda}\,\{p_i G_i(\mathbf{x}_0)\, E_0 + p_i H_i(\mathbf{x}_0) - b\,E_0^2 - kE_0 - l]. \quad ...(18)$$

4. Value of Second Variation

To decide whether a limiting optimal solution gives a local maximum, we examine the sign of

$$V_2 = \int_0^\infty \left[\left(\frac{\partial^2 H}{\partial x_i \partial x_j}\right)_0 \delta x_i \delta x_j + \left(\frac{\partial^2 H}{\partial x_i \partial E}\right)_0 \delta x_i \delta E + \frac{\partial^2 H}{\partial E^2}(\delta E)^2\right] dt. \quad ...(19)$$

Using (6), this gives

$$V_2 = \frac{1}{\lambda}\left\{p_k\left(\frac{\partial^2 G_k}{\partial x_i \partial x_j}\right)_0 E_0 + p_k\left(\frac{\partial^2 H_k}{\partial x_i \partial x_j}\right)_0 + C_k\left(\frac{\partial^2 F_k}{\partial x_i \partial x_j}\right)_0\right.$$

$$\left. - C_k E_0\left(\frac{\partial^2 G_k}{\partial x_i \partial x_j}\right)_0 - C_k\left(\frac{\partial^2 H_k}{\partial x_i \partial x_j}\right)\right\} \delta x_i \delta x_j$$

$$+ \frac{1}{\lambda}\left\{p_k\left(\frac{\partial G_k}{\partial x_i}\right)_0 - C_k\left(\frac{\partial G_k}{\partial x_i}\right)\right\} \delta x_i \delta E$$

$$- \frac{1}{\lambda}\, 2b(\delta E)^2. \quad ...(20)$$

For (16) to give a local maximum, V_2 should be negative definite.

5. Some Special Cases

(i) In the special case (11),

$$V_2 = -\frac{2b}{\lambda}(\delta E)^2 \qquad \ldots(21)$$

which is negative definite. In this case, the optimal solution gives a maximum.

(ii) If (11) is replaced by

$$F_i(\mathbf{x}) = a_i x_i - b_i x_i^2, \; G_i(\mathbf{x}) = \gamma_i, \; H_i(\mathbf{x}) = \alpha_i + \beta_{ij} x_j \qquad \ldots(22)$$

we get

$$V_2 = -\frac{2}{\lambda} b_k C_k (\delta x_i)(\delta x_i) - \frac{2b}{\lambda}(\delta E)^2 \qquad \ldots(23)$$

where from (14)

$$\begin{bmatrix} \beta_{11} + \lambda + a_1 & \beta_{21} & \cdots & \beta_{n1} \\ \beta_{12} & \beta_{22} + \lambda + a_2 \cdots & & \beta_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{1n} & \beta_{2n} & \cdots & \beta_{nn} + \lambda + a_n \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}. \qquad \ldots(24)$$

If the solution C's are positive, then V_2 is negative definite.

(iii) If in case (11) $\beta_{ii} = \beta_i$, $\beta_{ij} = 0$ when $i \neq j$, we get

$$C_j = \frac{p_j \beta_j}{\lambda + a_j + \beta_j} \qquad \ldots(25)$$

which gives V_2 as negative definite.

(iv) If

$$f_i = a_i + b_i x_i + f_{ijk} x_j x_k, \qquad \ldots(26)$$

then

$$V_2 = \frac{1}{\lambda} C_k f_{kij} \delta x_i \delta x_j - \frac{2b}{\lambda}(\delta E)^2. \qquad \ldots(27)$$

If $f_{ijk} x_j x_k$ is negative definite for each i and if C_k is given by (25), V_2 is again negative definite.

(v) For the two species case if

$$f_1 = (r_1 - a_1 x_1 - b_1 x_2)\, x_1, \; f_2 = (r_2 - a_2 x - b_2 x_2)\, x_2 \qquad \ldots(28)$$

$$V_2 = \frac{1}{\lambda}\left\{\frac{p_i\beta_1}{\lambda + r_1 + \beta_1}(-a_1(\delta x_1)^2 - b_1(\delta x_1 \delta x_2)) + \frac{p_2\beta_2}{\lambda + r_2 + \beta_2}(-a_2(\delta x_2)^2 - b_2(\delta x_1\delta x_2)\right\}. \quad \ldots(29)$$

In the case of competition, b_1, b_2 are both positive, but for predation, they can be of opposite sign V_2 will still be negative definite if

$$\frac{4p_1\beta_1 p_2\beta_2 a_1 a_2}{(\lambda_1 + r_1 + \beta_1)(\lambda_2 + r_2 + \beta_2)} \geqslant \left(\frac{p_1\beta_1 b_1}{\lambda + r_1 + \beta_1} + \frac{p_2\beta_2 b_2}{\lambda + r_2 + \beta_2}\right)^2. \quad \ldots(30)$$

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