REMARKS ON SOME FIXED POINT THEOREMS II

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The purpose of this note is to establish generalizations of a few fixed point theorems due to Chen and Shih (1976), Ciric (1977), Cheney and Goldstein (1959) and Reich (1972).

Recently Ciric (1977) proved the following theorem.

Theorem 1 — Let X be a closed convex subset of a normed space and let $T : X \to X$ satisfy the condition

$$d(Tx, Ty) \leqslant q \max \{ cd(x, y), [d(x, Tx) + d(y, Ty)], [d(x, Ty) + d(y, Tx)]\} \quad ...(1)$$

where $c \geqslant 0$, $0 \leqslant q < 1$. If a sequence $x_{n+1} = (1 - t)\, x_n + tx_n$, $n = 0, 1, 2, \ldots$, $x_0 \in X$, $0 < t < 1$, converges in X, then T has a fixed point.

We prove the following theorem.

Theorem 2 — Let X be a closed convex subset of a normed space and let T_1 and T_2 be two mappings of X into itself such that

$$d(T_1x, T_2y) \leqslant q \max \{cd(x, y), [d(x, T_1x) + d(y, T_2y)], [d(x, T_2y) + d(y, T_1x)]\} \quad ...(2)$$

where $c \geqslant 0$, $0 < q < 1$. If a sequence $\{x_n\}$ where $x_{2n+1} = (1 - t)\, x_{2n} + tT_1x_{2n}$, $x_{2(n+1)} = (1 - t)\, x_{2n+1} + tT_2x_{2n+1}$, $n = 0, 1, 2, \ldots$, $x_0 \in X$, $0 < t < 1$, converges in X, then T_1 and T_2 have a common fixed point.

PROOF : Let $u \in X$ such that $\lim \{x_n\} = u$. We show that u is a common fixed point of T_1 and T_2

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$$
\begin{aligned}
d(u, T_2u) &\leqslant d(u, x_{2n+1}) + \| (1-t)\, x_{2n} + tT_1x_{2n} - T_2u \| \\
&= d(u, x_{2n+1}) + \| (1-t)\, x_{2n} - (1-t)\, T_2u + tT_1x_{2n} - tT_2u \| \\
&\leqslant d(u, x_{2n+1}) + (1-t)\, d(x_{2n}, T_2u) + td(T_1x_{2n}, T_2u) \\
&\leqslant d(u, x_{2n+1}) + (1-t)\, d(x_{2n}, T_2u) \\
&\quad + tq \max \{cd(x_{2n}, u), [d(x_{2n}, T_1x_{2n}) + d(u, T_2u)], \\
&\qquad [d(x_{2n}, T_2u) + d(u, T_1x_{2n})]\}.
\end{aligned}
$$

Since $\{x_n\}$ converges to u, any subsequence of $\{x_n\}$ converges to u and since

$$
\begin{aligned}
d(x_{2n+1}, x_{2n}) &= \| x_{2n+1} - x_{2n} \| \\
&= \| (1-t)\, x_{2n} + tT_1x_{2n} - x_{2n} \| \\
&= td(T_1x_{2n}, x_{2n}),
\end{aligned}
$$

so proceeding to the limit $n \to \infty$ we obtain

$$
d(u, T_2u) \leqslant (1-t)\, d(u, T_2u) + tqd(u, T_2u),
$$

which implies $u = T_2u$. Similarly one can show that $u = T_1u$. Thus u is a common fixed point of T_1 and T_2.

This completes the proof of the theorem. Chen and Shih (1976) proved the following theorem.

Theorem 3 — Let X be a convex subset of a normed linear space. Let T be a self-map of X. Suppose that for all x, y in X

$$
\begin{aligned}
\| Tx - Ty \| \leqslant \max \{\| x - y \|, \tfrac{1}{2} [\| x - Tx \| + \| y - Ty \|], \\
\tfrac{1}{2} [\| x - Ty \| + \| y - Tx \|]\}. \quad \ldots(3)
\end{aligned}
$$

Let $x_0 \in X$, $t \in (0, 1)$ and $x_{n+1} = (1-t)\, x_n + tTx_n$, for each integer $n \geqslant 0$. Suppose that the sequence $\{x_n\}$ converges to a point u in X. Then u is a fixed point of T. The following theorem generalizes Theorem 3.

Theorem 4 — Let X be a convex subset of a normed linear space and let T_1 and T_2 be two self mappings of X such that for all x, y in X

$$
\begin{aligned}
\| T_1x - T_2y \| \leqslant \max \{\| x - y \|, \tfrac{1}{2} [\| x - T_1x \| + \| y - T_2y \|], \\
\tfrac{1}{2} [\| x - T_2y \| + \| y - T_1x \|]\}. \quad \ldots(4)
\end{aligned}
$$

Let $x_0 \in X$, $t \in (0, 1)$ and $x_{2n+1} = (1-t)\, x_{2n} + tT_1x_{2n}$, $x_{2(n+1)} = (1-t)\, x_{2n+1} + tT_2x_{2n+1}$ for each $n \geqslant 0$. If the sequence $\{x_n\}$ defined as above converges to a point $u \in X$ then u is the common fixed point of T_1 and T_2.

PROOF : We have

$$
\| x_{2n+1} - T_2u \| \leqslant (1-t)\, \| x_{2n} - T_2u \| + t\, \| T_1x_{2n} - T_2u \|. \quad \ldots(5)
$$

By (4) we get

$$\| T_1x_{2n} - T_2u \| \leqslant \max \{ \| x_{2n} - u \|, \tfrac{1}{2} [\| x_{2n} - T_1x_{2n} \| + \| u - T_2u \|], \tfrac{1}{2} [\| x_{2n} - T_2u \| + \| u - T_1x_{2n} \|]\}. \quad \ldots(6)$$

Hence from (5) and (6) we obtain

$$\| x_{2n+1} - T_2u \| \leqslant (1 - t) \| x_{2n} - T_2u \| + t \max \{\| x_{2n} - u \|, \tfrac{1}{2} [\| x_{2n} - T_1x_{2n} \| + \| u - T_2u \|], \tfrac{1}{2} [\| x_{2n} + T_2u \| + \| u - T_1x_{2n} \|]\}. \quad \ldots(7)$$

Letting $n \to \infty$ in (7) we get

$$\| u - T_2u \| \leqslant (1 - t) \| u - T_2u \| + \frac{t}{2} \| u - T_2u \| = \left(1 - \frac{t}{2}\right) \| u - T_2u \|,$$

which gives $u = T_2u$. Similarly $u = T_1u$. This completes the proof.

Cheney and Goldstein (1959) proved the following theorem.

Theorem 5 — Let T be a mapping of a metric space X into itself such that

(i) $d(Tx, Ty) \leqslant d(x, y)$;

(ii) if $x \neq Tx$, then $d(Tx, T^2x) < d(x, Tx)$;

(iii) for some $x_0 \in X$ the sequence $\{T^n x_0\}$ has a subsequence $\{T^{n_k} x_0\}$ converging to x. Then the sequence $\{T^n x_0\}$ converges to x and x is a fixed point of T.

Under a slightly stronger hypothesis, the following result generalizes Theorem 5.

Theorem 6 — Let (X, d) be a compact metric space, T be a self-mapping of X satisfying :

(iv) for each $x, y \in X$

$$d(Tx, Ty) \leqslant \max \{d(x, y), \tfrac{1}{2} [d(x, Tx) + d(y, Ty)], \tfrac{1}{2} [d(x, Ty) + d(y, Tx)]\};$$

(v) if $x \neq Tx$, then $d(Tx, T^2x) < d(x, Tx)$.

Then T has a fixed point in X.

PROOF : The proof follows the lines of proof of Theorem 1 in Chen and Shih (1976).

Following theorem generalizes Theorem 1 of Reich (1972).

Theorem 7 — Let (X, d) be a compact metric space and T be a self-mapping of X such that

(a) $d(Tx, Ty) \leqslant \max \{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)],$
$$\frac{1}{2}[d(x, Ty) + d(y, Tx)]\} \ \forall\ x, y \in X,$$

(b) $d(x, Tx)$ is not constant on any closed subset of X which contains more than one point and is invariant under T. Then T has a fixed point.

PROOF: We follow the same lines of proof as given by Reich (1972). Let $\mathcal{F}$ denote the collection of all non-empty closed subsets of X each of which is mapped into itself by T. Since X is a compact metric space, so by Zorn's lemma there exists a minimal element in $\mathcal{F}$ and let M be the minimal subset of X with respect to being non-empty, closed and invariant under T. If M contains more than point there are w and z with $r = d(w, Tw) < d(z, Tz)$.

Let $A = \{x \in M : d(x, Tx) \leqslant r$ and let $N =$ closure of $T(A)$. If $y \in N$ then y is the limit of $\{Tx_n\}$ with $d(x_n, Tx_n) \leqslant r$ for each n.

Now
$$\begin{aligned} d(y, Ty) &\leqslant d(y, Tx_n) + d(Tx_n, Ty) \\ &\leqslant d(y, Tx_n) + \max \{d(x_n, y), \tfrac{1}{2}[d(x_n, Tx_n) + d(y, Ty)], \\ &\qquad \tfrac{1}{2}[d(x_n, Ty) + d(y, Tx_n)]\} \\ &\leqslant d(y, Tx_n) + \max \{d(x_n, Tx_n) + d(y, Tx_n), \\ &\qquad \tfrac{1}{2}[d(x_n, Tx_n) + d(y, Ty)], \\ &\qquad \tfrac{1}{2}[d(x_n, Tx_n) + d(Tx_n, y) + d(y, Ty) + d(y, Tx_n)]\}. \end{aligned}$$

Proceeding to the limit $n \to \infty$ we obtain $d(y, Ty) \leqslant r$. Hence $N \subset A$ and $TN \subset TA \subset N$, which is a contradiction since N is a proper subset of M.

Next theorem generalizes Theorem 1 and Theorem 2 of Chen and Shih (1976)

Theorem 8 — Let T_1 and T_2 be two mappings of a compact metric space X into itself such that
$$d(T_1x, T_2y) < \max \{d(x, y), \tfrac{1}{2}[d(x, T_1x) + d(y, Ty)], \tfrac{1}{2}[d(x, T_2y) + d(y, T_1x)]\} \quad \ldots(8)$$
$$\forall\ x, y \in X,\ x \neq y.$$
Then T_1 and T_2 have a unique common fixed point.

PROOF: Suppose u_1 and u_2 are two distinct common fixed points of T_1 and T_2, then from (8) we obtain
$$d(u_1, u_2) = d(T_1u_1, T_2u_2) < \max \{d(u_1, u_2),\ 0,\ d(u_1, u_2)\}$$

i.e. $d(u_1, u_2) < d(u_1, u_2)$, a contradiction. So T_1 and T_2 can have at most one common fixed point.

Now let

$$q_i = \inf \{ d(x, T_i x)\} \text{ for } i = 1, 2.$$

Without any loss of generality, assume $r = q_2 \leqslant q_1$. Let $\{x_n\}$ be a sequence in X (where for arbitrary $x_0 \in X$, $x_{2n+1} = T_1 x_{2n}$, $x_{2(n+1)} = T_2 x_{2n+1}$, $n \geqslant 0$) such that $d(x_{2n+1}, T_2 x_{2n+1}) \to q_2 = r$. Since X is compact, there is a subsequence $\{y_{2(k+1)}\} = \{T_2 x_{2k+1}\} \subset \{x_n\}$, such that $\lim y_{2(k+1)} = y \in X$.

Then

$$\begin{aligned} d(y, T_1 y) &\leqslant d(y, y_{2(k+1)}) + d(T_1 y, T_2 x_{2k+1}) \\ &< d(y, y_{2k+2}) + \max \{d(y, x_{2k+1}), \\ &\qquad \tfrac{1}{2} [d(y, T_1 y) + d(x_{2k+1}, T_2 x_{2k+1})], \\ &\qquad \tfrac{1}{2} [d(y, T_2 x_{2k+1}) + d(x_{2k+1}, T_1 y)]\} \\ &\leqslant d(y, y_{2k+2}) + \max \{d(y, T_2 x_{2k+1}) + d(x_{2k+1}, T_2 x_{2k+1}), \\ &\qquad \tfrac{1}{2} [d(y, T_1 y) + d(x_{2k+1}, T_2 x_{2k+1})], \\ &\qquad \tfrac{1}{2} [d(y, T_2 x_{2k+1}) + d(x_{2k+1}, T_2 x_{2k+1}) \\ &\qquad\qquad + d(T_2 x_{2k+1}, y) + d(y, T_1 y)]\}. \end{aligned}$$

Proceeding to the limit $k \to \infty$ we obtain $d(y, T_1 y) \leqslant r \leqslant q_1$.

Hence we must have $r = q_2 = q_1 = d(y, T_1 y)$.

Suppose $r > 0$, then

$$\begin{aligned} &d(T_1 y, T_2 T_1 y) \\ &< \max \{d(y, T_1 y), \tfrac{1}{2} [d(y, T_1 y) + d(T_1 y, T_2 T_1 y)], \\ &\qquad \tfrac{1}{2} [d(y, T_2 T_1 y)]\} \end{aligned}$$

which gives

$$d(T_1 y, T_2 T_1 y) < d(y, T_1 y).$$

This contradiction completes the proof.

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