ON THE STRONG *SH* MOTION IN A TRANSVERSELY ISOTROPIC LAYER LYING OVER AN ISOTROPIC ELASTIC MATERIAL DUE TO A MOMENTARY POINT SOURCE

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In this paper the dispersion equation of Love waves in transversely isotropic layer over an isotropic elastic material due to a momentary point source has been found by the Green's function technique. As a special case the dispersion equation of Love waves when the layer is isotropic has been deduced. It has been shown that the displacement is very large when $\sin(h\sqrt{k_1^2 - f_n^2}) \to 0$ and this may cause large scale fracture in the free surface.

INTRODUCTION

Green's function has been discussed in detail by Friedman (1962). Covert (1958) has indicated a method of finding Green's function for composite bodies. Displacement of Love waves generated by a two dimentional point source in a layered medium has been studied by Sezawa (1935) and Satô (1952) by the method of successive reflections at the boundaries.

In this paper the same problem has been worked out by using Green's function technique. The method of reflections (cf. Feshbach and Morse 1953) has been used to find the usual dispersion equation of Love waves in transversely isotropic layer lying over an isotropic elastic material due to a momentary point source. This dispersion equation reduces to the usual dispersion equation with the layer to be isotropic when $L = N = \mu_1$, μ_1 being the modulus of rigidity in the layer and L, N are elastic constants. The displacement is very large when $\sin(h\sqrt{k_1^2 - f_n^2}) \to 0$, which may cause large scale fracture in the free surface.

SOLUTION OF THE PROBLEM

The layer has been assumed to be transversely isotropic of thickness h and the lower medium to be isotropic (Fig. 1). We take the origin on the interface of the layer denoted by medium 1 and the lower isotropic elastic medium denoted by medium 2. The x axis is taken horizontally along the interface and z-axis vertically downwards. The source has been taken in the medium 2 but immediately near the point $(x_0, 0)$.

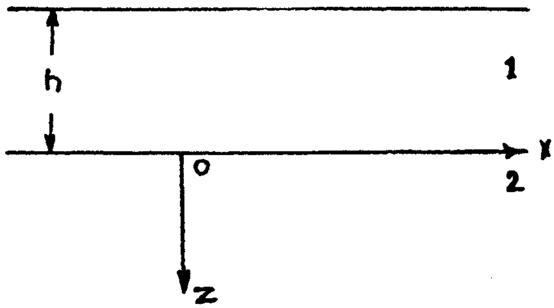

FIG. 1.

The equation of the transverse horizontal displacement in the medium 1, which is transversely isotropic is

$$\nabla_1^2 V = \frac{\rho}{N} \frac{\partial^2 V}{\partial t^2} \qquad \ldots(1)$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_1^2}$$

and

$$z = \sqrt{\frac{L}{N}} \cdot z_1 \qquad \ldots(2)$$

and for the lower isotropic medium the equation is

$$\nabla^2 V = \frac{\rho_2}{\mu} \frac{\partial^2 V}{\partial t^2} \qquad \ldots(3)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \qquad \ldots(4)$$

Taking the time dependent proportional to $e^{i\omega t}$, equations for media 1 and 2 are

$$\nabla_1^2 V_1 + k_1^2 V_1 = 0 \qquad \ldots(5)$$

and

$$\nabla^2 V_2 + k_2^2 V_2 = 0 \qquad \ldots(6)$$

where

$$\left.\begin{aligned} k_1^2 &= \frac{\omega^2 \rho}{N} \\ \text{and} \quad k_2^2 &= \frac{\omega^2 \rho_2}{\mu}. \end{aligned}\right\} \qquad \ldots(7)$$

The boundary conditions are

$$L \frac{\partial V_1}{\partial z} = 0 \quad \text{at} \quad z = -h \qquad ...(8)$$

$$V_1 = V_2 \qquad ...(9)$$

and

$$L \frac{\partial V_1}{\partial z} = \mu \frac{\partial V_2}{\partial z} \text{ at } z = 0. \qquad ...(10)$$

Covert (1958) indicated a method of finding Green's function for composite bodies.

Let G_1 and G_2 be the Green's function for bodies 1 and 2 under the boundary conditions $\frac{\partial G_1}{\partial n_1} = 0$, $\frac{\partial G_2}{\partial n_2} = 0$ at the interface and $G_1 = 0$ at the free surface; n_1, n_2 are normal drawn outwards from the region. Then we have

$$V_1(r) = \int G_1(r/r_0)\, \rho_1'(r_0)\, dv_1 + \frac{1}{4\pi} \int_{AB} G_1(r/r_s) \frac{\partial V_1(r_s)}{\partial n_1} ds_1 \qquad ...(11)$$

$$V_2(r) = \int G_2(r/r_0)\, \rho_2'(r_0)\, dv_2 + \frac{1}{4\pi} \int_{AB} G_2(r/r_s) \frac{\partial V_2(r_s)}{\partial n_2} ds_2 \qquad ...(12)$$

where ρ_1', ρ_2' are source densities in media 1 and 2. If the point source lies very near the origin but in the medium 2, that is, if $\rho_2' = \delta(r_0 - 0)$ and $\rho_1' = 0$, we have V_1 as Green's function for body 1 and V_2 as Green's function for body 2 when the source is in the medium 2.

From (9), (11) and (12) we have

$$\int G_1\rho_1'(r_0)\, dv_1 + \frac{1}{4\pi} \int_{AB} G_1 \frac{\partial V_1}{\partial n_1} ds_1$$

$$= \int G_2\rho_2'\, dv_2 + \frac{1}{4\pi} \int_{AB} G_2 \frac{\partial V_2}{\partial n_2} ds_2.$$

Using (10) and $\rho_1' = 0$, $\rho_2' = \delta(r_0 - 0)$, the above equation becomes

$$G_2(r/0) = \frac{1}{4\pi} \int_{AB} \left[G_1(r/r_0) + \frac{L}{\mu} G_2(r/r_0) \right] \frac{\partial G(0/r_0)}{\partial z} ds. \qquad ...(13)$$

where the field point $r(x, y)$ and variable point $r_0(x_0, 0)$ both are on the interface AB and G is the proper Green's function for body 1 corresponding to the source in the medium 2.

Under the boundary condition of Green's function we have

$$G_2(r/r_0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\exp(\alpha_2 z + if(x - x_0))}{\alpha_2} df \qquad ...(14)$$

where

$$\alpha_2^2 = f^2 - k_2^2. \qquad ...(15)$$

For the calculation of $G_1(r/r_0)$, we follow the reflection method. The reflected points are $(x_0, -2h)$, $(x_0, 2h)$, $(x_0, -4h)$, $(x_0, 4h)$, $(x_0, -6h)$, $(x_0, 6h)$, ...

Then,

$$\begin{aligned} G_1(r/r_0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \Bigg[& \exp\left(\sqrt{\frac{N}{L}}\alpha z + if(x - x_0)\right) \\ & + \exp\left(-\alpha\left(\sqrt{\frac{N}{L}}\alpha z + 2h\right) + if(x - x_0)\right) \\ & + \exp\left(-\alpha\left(2h - \sqrt{\frac{N}{L}} z\right) + if(x - x_0)\right) \\ & + \exp\left(-\alpha\left(\sqrt{\frac{N}{L}} z + 4h\right) + if(x - x_0)\right) \\ & + \exp\left(-\alpha\left(4h - \sqrt{\frac{N}{L}} z\right) + if(x - x_0)\right) \\ & + \exp\left(-\alpha\left(\sqrt{\frac{N}{L}} z + 6h\right) + if(x - x_0)\right) + \dots \Bigg] \frac{df}{\alpha} \end{aligned} \qquad ...(16)$$

where

$$\alpha^2 = f^2 - k_1^2 \qquad ...(17)$$

or,

$$G_1(r/r_0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\exp\left(\alpha\sqrt{\frac{N}{L}} z\right) + \exp\left(-\alpha\left(\sqrt{\frac{N}{L}} z + 2h\right)\right)}{1 - e^{-2h\alpha}} \times \frac{\exp(if(x - x_0))}{\alpha} df. \qquad ...(18)$$

From (11), we have,

$$G(r/0) = \frac{1}{4\pi} \int\limits_{AB} G_1 \frac{\partial G}{\partial z}\, ds$$

which is the expression for Green's function for the body 1 corresponding to the source in the body 2. Taking the contour AB on the x axis from $-\infty$ to ∞, we have

$$G(r/0) = \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} G_1 \frac{\partial G(0/r_0)}{\partial z}\, dx_0. \qquad ...(19)$$

Using the relation (13), (14) and (18) we get

$$\frac{2}{\pi} \int\limits_{-\infty}^{\infty} \frac{e^{ifx}}{\alpha_2}\, df = \frac{1}{4\pi} \int\limits_{AB} \left[\frac{2}{\pi} \int\limits_{-\infty}^{\infty} \frac{1 + e^{-2h\alpha}}{1 - e^{-2h\alpha}} \cdot \frac{e^{if(x-x_0)}}{\alpha}\, df \right.$$

$$\left. + \frac{2}{\pi} \cdot \frac{L}{\mu} \int\limits_{-\infty}^{\infty} \frac{e^{if(x-x_0)}}{\alpha_2}\, df \right] \frac{\partial G(0/r_0)}{\partial z}\, ds.$$

Writing the integral along AB as $-\infty$ to ∞ :

$$\frac{1}{\alpha_2} = \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} \left[\frac{L}{\mu\alpha_2} + \frac{1 + e^{-2h\alpha}}{\alpha(1 - e^{-2h\alpha})} \right] e^{ifx_0} \frac{\partial G(0/r_0)}{\partial z}\, dx_0.$$

Now, by applying Fourier inversion theorem, we have

$$\frac{\partial G(0/r_0)}{\partial z} = 2 \int\limits_{-\infty}^{\infty} \frac{e^{+ifx_0}\, df}{\alpha_2 \left[\frac{L}{\mu\alpha_2} + \frac{1 + e^{-2h\alpha}}{\alpha(1 - e^{-2h\alpha})} \right]}. \qquad ...(20)$$

Substituting the value of $\partial G(0/r_0)/\partial z$ and G_1 in the eqn. (19), we have

$$G(r/0) = \frac{1}{\pi^2} \int\limits_{-\infty}^{\infty} dx_0 \left[\iint\limits_{-\infty}^{\infty} \frac{\exp\left(\alpha \sqrt{\frac{N}{L}}\, z\right) + \exp\left(-\alpha \left(\sqrt{\frac{N}{L}}\, z + 2h\right)\right)}{\alpha(1 - e^{-2h\alpha})} \right.$$

$$\left. \times \frac{e^{if'x_0} \cdot e^{if(x-x_0)}}{\alpha_2' \left[\frac{L'}{\mu\alpha_2'} + \frac{1 + e^{-2h\alpha'}}{\alpha'(1 - e^{-2h\alpha'})} \right]}\, df\, df' \right]. \qquad ...(21)$$

Put $f' - f = \eta$, therefore $df' = d\eta$ and using the result

$$\delta(f' - f) \doteq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(f'-f)x_0}\, dx_0$$

and

$$\int g(f')\, \delta(f' - f)\, df' = g(f),$$

eqn. (21) becomes

$$G(r/0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\exp\left(\alpha \sqrt{\frac{N}{L}}\, z\right) + \exp\left(-\alpha\left(\sqrt{\frac{N}{L}}\, z + 2h\right)\right)}{1 - e^{-2h\alpha}}$$

$$\times \frac{e^{ifx} df}{\alpha\alpha_2\left[\frac{L}{\mu\alpha_2} + \frac{1 + e^{-2h\alpha}}{\alpha(1 - e^{-2h\alpha})}\right]}. \qquad \ldots(22)$$

It is the expression for *SH* displacement at a point corresponding to a source in the lower medium but at the origin (very nearby). In order to evaluate the integral we choose the contour as real axis and infinite semi-circle in the upper half plane with necessary cuts at the branch points $f = k_1, k_2$. We need consider only the contribution from the poles for surface *SH* motion. The poles of $G(r/0)$ are given by the roots of

$$\frac{L}{\mu\alpha_2} + \frac{1 + e^{-2h\alpha}}{\alpha(1 - e^{-2h\alpha})} = 0. \qquad \ldots(23)$$

From (15) and (17) we have,

$$\alpha = i\sqrt{\omega^2 \frac{\rho}{N} - f^2}$$

and

$$\alpha_2 = \sqrt{f^2 - \frac{\omega^2 \rho_2}{\mu}}$$

then (23) becomes

$$\tan\left(h\sqrt{\omega^2 \frac{\rho}{N} - f^2}\right) = \frac{\mu}{L} \cdot \frac{\sqrt{f^2 - \frac{\omega^2\rho_2}{\mu}}}{\sqrt{\omega^2 \frac{\rho}{N} - f^2}}$$

which is the frequency equation for Love waves in transversely isotropic layer. If the upper layer is isotropic, then $L = N = \mu_1$, (say), which leads to the usual dispersion equation of Love waves.

Discussion

Equation (22) can be written as

$$G(r/0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\exp\left(\alpha\left(\sqrt{\frac{N}{L}}z + h\right)\right) + \exp\left(-\alpha\left(\sqrt{\frac{N}{L}}z + h\right)\right)}{e^{\alpha h} - e^{-\alpha h}}$$

$$\times \frac{e^{ifx}}{\alpha\left\{\frac{L}{\mu} + \frac{\alpha_2(1 + e^{-2h\alpha})}{\alpha(1 - e^{-2h\alpha})}\right\}} df.$$

The contribution due to poles at $z = -h$, i.e. at the surface is

$$2\pi i \cdot \frac{2}{\pi} \sum \frac{\exp\left(\alpha_n\left(-\sqrt{\frac{N}{L}} + 1\right)h\right) + \exp\left(-\alpha_n\left(-\sqrt{\frac{N}{L}} + 1\right)h\right)}{\alpha_n(e^{h\alpha_n} - e^{-h\alpha_n})\, F'(f_n)} \qquad \ldots(24)$$

where

$$F(f) = \frac{L}{\mu} + \frac{\alpha_2(1 + e^{-2h\alpha})}{\alpha(1 - e^{-2h\alpha})}$$

f_n is a root of $F(f) = 0$ and α_n is of α in which f_n is put for f. Now in (24), when

$$e^{\alpha_n h} - e^{-\alpha_n h} \to 0, \quad \text{where } \alpha_n = i(k_1^2 - f_n^2)^{1/2}$$

we have,

$$\sin(h\sqrt{k_1^2 - f_n^2}) \to 0$$

i.e. $$h\sqrt{k_1^2 - f_n^2} = m\pi$$

and the amplitude expression of (22) is very large and this may cause fracture.

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